Introduction to Applied Math

Homework 2

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Complete the following exercises. Show **all** your work and justify your answers. **Unjustified answer will not be evaluated.** You may use any method you have learned in class.

Basic Exercises

1. Consider the following system of linear equations in three variables:

$$\begin{cases} x + 2y + z = 3\,, \\ 2x + 5y - z = -4\,, \\ 3x - 2y + 4z = 20\,. \end{cases}$$

(a) Write down the augmented matrix $[A \mid \mathbf{b}]$ corresponding to this linear system.

(b) Apply elementary row operations to reduce the augmented matrix to row-echelon form. State each row operation performed clearly.

(c) Use back-substitution to determine the unique solution (x, y, z) of the system.

(d) Provide a geometric interpretation of the solution in $\mathbb{R}^3$ in terms of the planes represented by each equation.

2. Consider the following underdetermined system of linear equations in four variables x_1, x_2, x_3, x_4:

$$\begin{cases} x_1 - 2x_2 + x_3 + 2x_4 = 1\,, \\ 2x_1 - 4x_2 + 3x_3 + 3x_4 = 3\,, \\ -x_1 + 2x_2 - 2x_3 - x_4 = -2\,. \end{cases}$$

(a) Use Gauss–Jordan elimination to transform the augmented matrix of the system into reduced row-echelon form.

(b) Identify the pivot columns and state which variables are leading (basic) variables and which are free variables.

(c) Express the general solution in parametric vector form, identifying a particular solution $\mathbf{x}_p$ and the general solution $\mathbf{x}_h$ to the associated homogeneous system.

3. Consider the matrices:

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 \\ 0 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & 1 \end{bmatrix}.$$

(a) Compute the linear combination $2A - 3B$.

(b) Compute the matrix products AB and BA. Does $AB = BA$? What does this show about matrix multiplication?

(c) Compute $(AB)^T$ and verify the identity $(AB)^T = B^T A^T$ by computing $B^T A^T$ explicitly.

(d) Determine whether the matrix products AC and CA are defined. If defined, compute the product; if not defined, explain why in terms of matrix dimensions.

4. (a) Evaluate the determinant of the following 4×4 matrix by using elementary row operations to introduce zeros, or by cofactor expansion:

$$M = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 3 & 1 & -2 \\ 2 & 4 & 1 & 5 \\ -1 & -2 & 3 & -1 \end{bmatrix}.$$

(b) Suppose A and B are 3×3 matrices such that $\det(A) = 4$ and $\det(B) = -2$. Using the algebraic properties of determinants (without determining the matrices A and B), compute the exact value of each of the following:

(i) $\det(AB)$

(ii) $\det(2A)$

(iii) $\det(A^T B^2)$

(iv) $\det(-A)$

(c) Based on your result in part (b)(i), is the product matrix AB invertible? Justify your answer using the determinant criterion.

5. Consider the following three vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

(a) Prove that the set $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ forms a basis for $\mathbb{R}^3$.

(b) Consider the two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ from this basis, and let W be the space defined by their linear combinations:

$$W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\} = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 \mid c_1, c_2 \in \mathbb{R}\}.$$

Prove directly, using the **Subspace Test** (checking that $\mathbf{0} \in W$, closure under vector addition, and closure under scalar multiplication), that W is a subspace of $\mathbb{R}^3$ (and therefore is a vector space).

(c) Now, provide an alternative proof of part (b) by using the criterion that the solution set of a homogeneous linear system describes a subspace of $\mathbb{R}^3$ (**Theorem 2, page 226** of the textbook).
Hint: Find a non-zero normal vector $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$ to determine the single Cartesian equation $ax + by + cz = 0$ of the plane spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$. Then express W as the set of solutions to a matrix equation $A\mathbf{x} = \mathbf{0}$ for a 1×3 matrix A, and invoke Theorem 2.

Challenge Exercises

You must complete at least one of the following exercises. You may attempt more than one if you wish.

1. (a) State the **Subspace Test**: What conditions must a subset $W \subseteq \mathbb{R}^n$ satisfy in order to be a subspace of $\mathbb{R}^n$?

(b) Determine whether each of the following subsets of $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$. If it is a subspace, prove it by verifying all conditions of the subspace test. If it is not a subspace, provide an explicit numerical counterexample:

(i) $W_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 3x_1 - 4x_2 + 2x_3 = 0\}$
(ii) $W_2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 - x_2 + 2x_3 = 5\}$
(iii) $W_3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 x_2 \geq 0\}$

2. Consider the following system of linear equations depending on a real parameter $k \in \mathbb{R}$:

$$\begin{cases} x + y + kz = 1\,, \\ x + ky + z = 1\,, \\ kx + y + z = 1\,. \end{cases}$$

(a) Let A be the coefficient matrix of the system. Compute the determinant $\det(A)$ and express it in fully factored form in terms of k.

(b) Using the determinant and row reduction, determine all values of k for which the system has:

(i) A unique solution. (Also find the unique solution in terms of k).
(ii) Infinitely many solutions.
(iii) No solution (the system is inconsistent).

(c) For the value(s) of k where the system has infinitely many solutions, find the general solution and express it in parametric vector form.