Complex Analysis

Homework 1

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Undergraduate Exercises

Complete the following exercises:

1. Use De Moivre's formula to prove the following identities:

$$\cos(3\theta) = \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta)\,, \quad \sin(3\theta) = 3\cos^2(\theta)\sin(\theta) - \sin^3(\theta)\,.$$

2. Sketch the following sets in the complex plane:

$$A = \{z \in \mathbb{C} \mid |z - 1 + i| = 2\}\,, \quad B = \{z \in \mathbb{C} \mid \Re(z^2) > -2\Im(z) + 2\}\,.$$

3. Compute the following powers of complex numbers:

$$(1+i)^{13}\,, \quad \left(\frac{1 - i\sqrt{3}}{2}\right)^{10}\,.$$

4. Find the principal values of the arguments, $\mathrm{Arg}(z)$, for the following complex numbers:

$$z_1 = -\sqrt{3} - i\,, \quad z_2 = (1+i)^5\,, \quad z_3 = \frac{1 - i\sqrt{3}}{2}\,, \quad z_4 = \frac{2i}{1 + \sqrt{3}i}\,.$$

5. Find all complex solutions to the equation:

$$z^4 + 16 = 0\,.$$

Use these solutions to rewrite the polynomial $P(z) = z^4 + 16$ as a product of two polynomials of degree 2 with real coefficients.

Graduate Exercises

Graduate students must complete all undergraduate exercises plus the following additional problems:

1. Consider the geometric sum formula for $z \neq 1$:

$$\sum_{k=0}^{n} z^k = \frac{1 - z^{n+1}}{1 - z} \,.$$

Use this result to prove Lagrange's trigonometric identities:

$$\sum_{k=0}^{n} \cos(k\theta) = \frac{1}{2} + \frac{\sin\left(\left(n + \frac{1}{2}\right)\theta\right)}{2\sin\left(\frac{\theta}{2}\right)} \,, \qquad \sum_{k=0}^{n} \sin(k\theta) = \frac{\cos\left(\frac{\theta}{2}\right) - \cos\left(\left(n + \frac{1}{2}\right)\theta\right)}{2\sin\left(\frac{\theta}{2}\right)} \,.$$

Hint: Use Euler's formula to express the trigonometric functions in terms of complex exponentials, use the $\Re(z)$ and $\Im(z)$ operators and use the sum-to-product formula.

2. Using the Taylor series expansion, prove that:

$$e^{iz} = \cos(z) + i\sin(z) \,,$$

holds for any complex number $z \in \mathbb{C}$.