Complex Analysis

Homework 2

Guido Mazzuca

February 9, 2026

Undergraduate Exercises

Complete the following exercises:

1. Compute the following limits.

$$\lim_{z\to -i} \frac{z+i}{z^2+(i+1)z+i}\,, \quad \lim_{z\to\infty} \frac{z^3}{z^4+2iz+3}\,.$$

2. Assume that $f(z) = u(x,y)+iv(x,y)$ is a differentiable function in a domain $D \subseteq \mathbb{C}$. Show that if there exists a point $z_0 \in D$ such that $f'(z_0) = 0$, then both $u(x,y)$ and $v(x,y)$ have a critical point at (x_0, y_0) (where $z_0 = x_0 + iy_0$).

3. Show that if $v(x,y)$ is the harmonic conjugate of $u(x,y)$ in a domain D, and $u(x,y)$ is simultaneously the harmonic conjugate of $v(x,y)$ in D, then both $u(x,y)$ and $v(x,y)$ must be constant functions.

4. Assume that $f(z) = u(r,\theta) + iv(r,\theta)$ is an analytic function in a domain $D \subseteq \mathbb{C}$ that does not contain the origin $(0,0)$. Show that $u(r,\theta)$ satisfies Laplace's equation in polar coordinates:

$$r^2 u_{rr} + r u_r + u_{\theta\theta} = 0\,.$$

Verify that the same equation holds for $v(r,\theta)$.

5. Using the principal branch of the logarithm (Log), show that:

$$\operatorname{Log}((1+i)^2) = 2\operatorname{Log}(1+i)$$

but

$$\operatorname{Log}((-1+i)^2) \neq 2\operatorname{Log}(-1+i)$$

6. (**A sprinkle of Guido's research**) Using the principal branch of the logarithm, show that:

$$\lim_{\varepsilon\to 0^+} [\operatorname{Log}(-1+i\varepsilon) - \operatorname{Log}(-1-i\varepsilon)] = 2\pi i\,.$$

Graduate Exercises

Graduate students must complete all undergraduate exercises plus the following additional problems:

1. Determine the branch cuts for the principal branches of the following functions:

$$f(z) = \mathrm{Log}(z+i) - \mathrm{Log}(z-i)\,, \quad g(z) = \mathrm{Log}\left(\frac{z+i}{z-i}\right)\,.$$

Explain why the domains of analyticity for $f(z)$ and $g(z)$ are different.

2. (**A second sprinkle of Guido's research**) Using the principal branch of the square root function, show that the functions:

$$R(z) = (z-1)^{1/2}(z+1)^{1/2}\,, \quad S(z) = ((z-1)(z+1))^{1/2}\,,$$

have different branch cuts in the complex plane.