Complex Analysis

Homework 3: Integrals, Cauchy's Theorems, and Laurent Series

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Undergraduate Exercises

Complete the following exercises. Please justify your steps and explicitly state when you are invoking a specific theorem (e.g., Cauchy-Goursat, Cauchy Integral Formula, etc.).

1. **Parametrization and Basic Integration.**
   Evaluate the contour integral
   $$\int_C \mathrm{Re}(z)\, dz$$
   where C is the straight line segment from $z_1 = 0$ to $z_2 = 1 + i$, followed by the straight line segment from $z_2 = 1 + i$ to $z_3 = 2i$.

2. **Cauchy Integral Formula.**
   Evaluate the following integrals over the given simple closed contours, assuming counterclockwise orientation.

   (a) $\displaystyle\oint_{|z|=2} \frac{e^{\pi z}}{z(z^2+9)}\, dz$

   (b) $\displaystyle\oint_{|z-i|=1.5} \frac{\sin(z)}{(z-i)^3}\, dz$

3. **Liouville's Theorem.**
   Let $f(z)$ be an entire function. Suppose there exists a positive real constant M such that $\mathrm{Re}(f(z)) \leq M$ for all $z \in \mathbb{C}$. By considering the new entire function $g(z) = e^{f(z)}$, use Liouville's Theorem to prove that $f(z)$ must be a constant function.

4. **Laurent Series in Different Regions.**
   Consider the rational function:
   $$f(z) = \frac{3}{z^2 + z - 2}$$
   Find the Laurent series expansion for $f(z)$ centered at $z_0 = 0$ in each of the following domains:

   (a) The disk $|z| < 1$.

(b) The annulus $1 < |z| < 2$.

(c) The region $|z| > 2$.

Hint: Start by using partial fraction decomposition on $f(z)$.

Graduate Exercises

Graduate students must complete all undergraduate exercises plus the following additional problems:

1. **The Fundamental Theorem of Algebra.**
Use Liouville's Theorem to prove the Fundamental Theorem of Algebra: *Every non-constant polynomial $P(z)$ with complex coefficients has at least one root in $\mathbb{C}$.*
Hint: Proceed by contradiction. Assume $P(z)$ has no roots in $\mathbb{C}$. What can you say about the function $f(z) = \frac{1}{P(z)}$? Show that it is entire and bounded.

2. **Laurent Expansion around a Singularity.**
Find the Laurent series expansion of the function

$$h(z) = \frac{1}{z(z-2)}$$

valid in the punctured disk (annulus) $0 < |z-2| < 2$.
Hint: You are expanding around $z_0 = 2$. It is highly recommended to use the substitution $w = z - 2$, expand the resulting function in terms of w, and then substitute back.