Complex Analysis

Homework 4: Residues and Contour Integration

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Undergraduate Exercises

Complete the following exercises. Please justify your steps and explicitly state when you are invoking a specific theorem (e.g., Cauchy-Goursat, Cauchy Integral Formula, etc.).

Exercise 1 (Laurent Series and High Derivatives)**.** Consider the rational function:

$$f(z) = \frac{1}{(1+z^2)(z-3)}$$

(a) Find the partial fraction decomposition of $f(z)$.

(b) Compute the Laurent series expansion for $f(z)$ centered at $z_0 = 0$ in the following three domains:

i. $|z| < 1$

ii. $1 < |z| < 3$

iii. $|z| > 3$

(c) Using your result from the region where $f(z)$ is analytic at the origin (the Maclaurin series), compute the 2026^{th} derivative of $f(z)$ evaluated at $z = 0$. That is, find $f^{(2026)}(0)$. *Hint: Do not try to differentiate the function 2026 times.*

Exercise 2 (Computing Integrals via Residues)**.** Use Cauchy's Residue Theorem to evaluate the following contour integrals. In each case, C is a simple closed contour traversed in the counterclockwise direction.

(a) $\displaystyle\oint_{|z|=2} \frac{e^{\pi z}}{z^2(z^2+9)}\, dz$

(b) $\displaystyle\oint_{|z|=4} \frac{\cos(z)}{z^2(z-\pi)^3}\, dz$

Exercise 3 (Generating Series via Differentiation and Integration)**.** Instead of computing derivatives, we can generate new power series by integrating or differentiating known Maclaurin series term-by-term.

(a) **By Integration:** Compute the power series expansion of $f(z) = \arctan(z)$ centered at $z_0 = 0$. What is its radius of convergence?

Hint: take the derivative and expand the derivative in power series

(b) **By Differentiation:** Compute the power series of $g(z) = \frac{z^2}{(1-z^3)^2}$ centered at $z_0 = 0$. *Hint: Integrate and expand the primitive in power series*

Exercise 4 (Evaluating Real Integrals)**.** Use the Residue Theorem to evaluate the following improper integral along the real line:

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\, dx$$

Hint: Consider a semicircular contour in the upper half-plane. Justify why the integral over the circular arc vanishes as $R \to \infty$.

Graduate Exercises

Graduate students must complete all undergraduate exercises plus the following additional problems:

Exercise 5 (Integration Involving Branch cuts)**.** Evaluate the following integral by residue:

$$\int_0^1 \frac{1}{\sqrt{x(1-x)}\sqrt{(2-x)(3-x)}}\, dx - \int_2^3 \frac{1}{\sqrt{x(x-1)}\sqrt{(x-2)(3-x)}}\, dx$$

Hint: Reduce the following integral to the integral of the complex function $f(z) = \frac{1}{z^{1/2}(z-1)^{1/2}(z-2)^{1/2}(z-3)^{1/2}}$ over a circle of radius $R \gg 3$

Exercise 6 (Summing Series with Residues)**.** The Residue Theorem provides a powerful method for evaluating infinite sums by integrating meromorphic functions over large square contours C_N.

(a) Let $a > 0$ be a real number. Consider the function $f(z) = \frac{\pi \cot(\pi z)}{z^2+a^2}$. Find all poles of $f(z)$ in the complex plane and compute the residue at each pole.

(b) By integrating $f(z)$ over a large square contour C_N centered at the origin with vertices at $\pm(N+\frac{1}{2}) \pm i(N+\frac{1}{2})$ and letting $N \to \infty$, prove the following summation formula:

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{a}\coth(\pi a)$$

Note: You should prove that the integral over C_N goes to 0 as $N \to \infty$ and then use the residue theorem.