Lecture Notes on Complex Numbers

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January 20, 2026

Contents

Chapter 1

Complex Numbers

1.1 Sums and Products

The complex numbers can be defined as ordered pairs of real numbers (x, y) subject to specific operations of addition and multiplication. We identify the set of complex numbers $\mathbb{C}$ with the xy-plane (the complex plane).

Definition 1.1.1 (Complex Number)**.** A complex number z is defined as an ordered pair of real numbers:

$$z = (x, y)$$

where x is the *real part* of z, denoted $\text{Re}(z)$, and y is the *imaginary part* of z, denoted $\text{Im}(z)$.

Two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are equal if and only if their real parts are equal and their imaginary parts are equal.

Definition 1.1.2 (Algebraic Operations)**.** Let $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$. We define sum and product as:

$$\begin{aligned} z_1 + z_2 &= (x_1 + x_2, y_1 + y_2) \\ z_1 z_2 &= (x_1 x_2 - y_1 y_2, y_1 x_2 + x_1 y_2) \end{aligned}$$

Remark 1.1.1. The set of complex numbers contains the set of real numbers as a subset. We identify the real number x with the pair $(x, 0)$. Under this identification, operations correspond to standard real arithmetic.

1.1.1 The Imaginary Unit

It is customary to denote the number $(0, 1)$ by i. Using the definition of multiplication:

$$i^2 = (0, 1)(0, 1) = (0 \cdot 0 - 1 \cdot 1, 1 \cdot 0 + 0 \cdot 1) = (-1, 0) = -1$$

Thus, we can write any complex number $z = (x, y)$ as:

$$z = x + iy$$

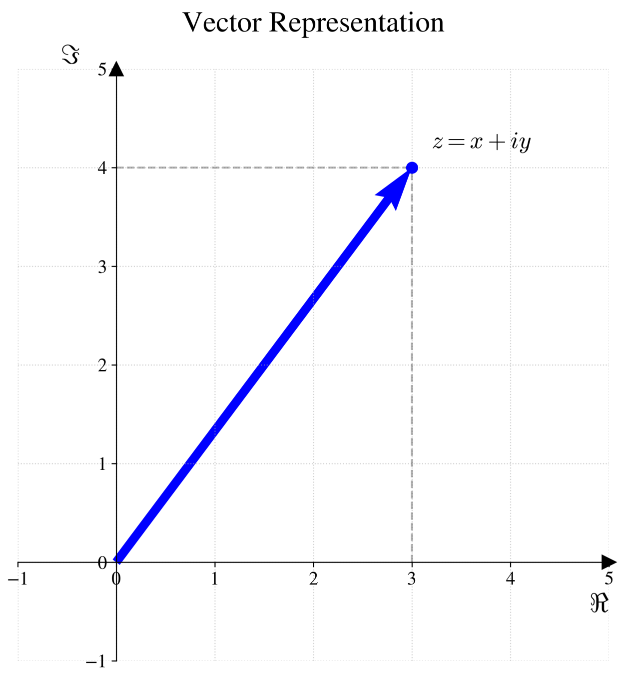

Figure 1.1: Vector representation of a complex number $z = x + iy$. The point (x, y) corresponds to the complex number in the plane.

1.2 Basic Algebraic Properties

The set $\mathbb{C}$ forms a **field** under addition and multiplication.

Theorem 1.2.1 (Field Properties)**.** *For any $z_1, z_2, z_3 \in \mathbb{C}$, the following laws hold:*

1. ***Commutative laws:*** $z_1 + z_2 = z_2 + z_1$ *and* $z_1 z_2 = z_2 z_1$.
2. ***Associative laws:*** $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ *and* $(z_1 z_2) z_3 = z_1(z_2 z_3)$.
3. ***Distributive law:*** $z(z_1 + z_2) = zz_1 + zz_2$.
4. ***Identities:*** *Additive identity is* $0 = (0, 0)$. *Multiplicative identity is* $1 = (1, 0)$.
5. ***Inverses:*** *For every z, there is an additive inverse $-z$. For every $z \neq 0$, there is a multiplicative inverse z^{-1}.*

Example 1.2.1. Find the multiplicative inverse of $z = x + iy$ assuming $z \neq 0$. We seek $u + iv$ such that $(x + iy)(u + iv) = 1$. Solving the system of linear equations yields:

$$z^{-1} = \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right)$$

1.3 Vectors and Moduli

A complex number $z = x + iy$ is naturally associated with a position vector in the plane starting from the origin and terminating at (x, y).

Definition 1.3.1 (Modulus)**.** The modulus (or absolute value) of $z = x + iy$ is the non-negative real number:

$$|z| = \sqrt{x^2 + y^2}$$

Geometrically, this represents the distance of the point z from the origin.

Lemma 1.3.1 (Distance between points)**.** *The distance between two complex numbers z_1 and z_2 is given by $|z_1 - z_2|$.*

Exercise 1.3.1. Prove that $|z|^2 = (\mathrm{Re}(z))^2 + (\mathrm{Im}(z))^2$.

1.4 The Triangle Inequality

One of the most crucial inequalities in analysis is the Triangle Inequality, which states that the length of a side of a triangle is less than or equal to the sum of the lengths of the other two sides.

Theorem 1.4.1 (Triangle Inequality)**.** *For any $z_1, z_2 \in \mathbb{C}$:*

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

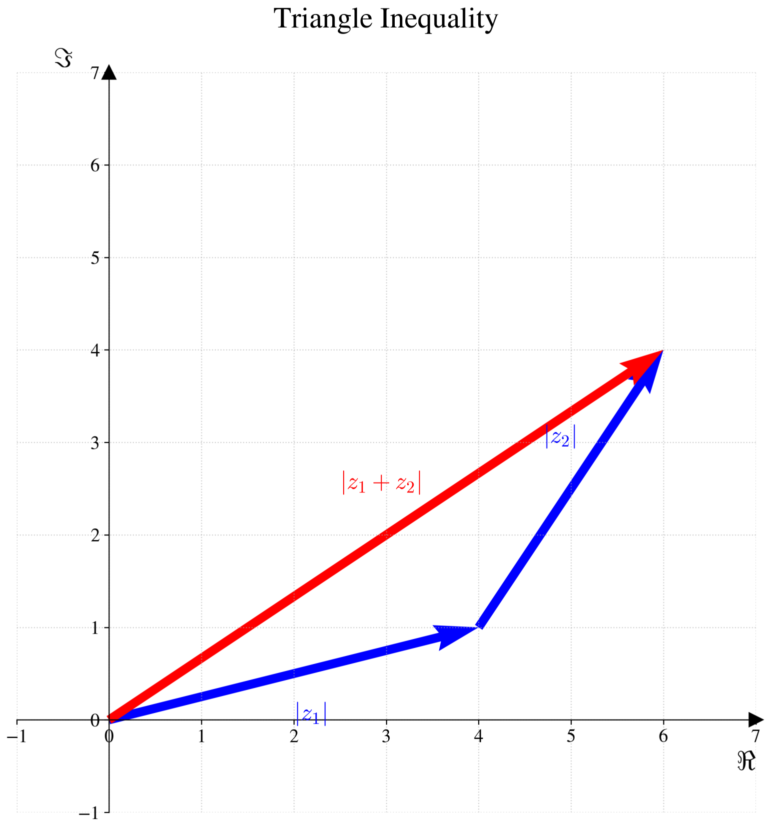

Figure 1.2: Geometric representation of the Triangle Inequality for complex numbers z_1 and z_2. The length of the vector $z_1 + z_2$ is less than or equal to the sum of the lengths of z_1 and z_2.

Corollary 1.4.2. *It follows from the theorem that:*

$$|z_1 + z_2| \geq ||z_1| - |z_2||$$

See Figure 1.2 for a geometric illustration.

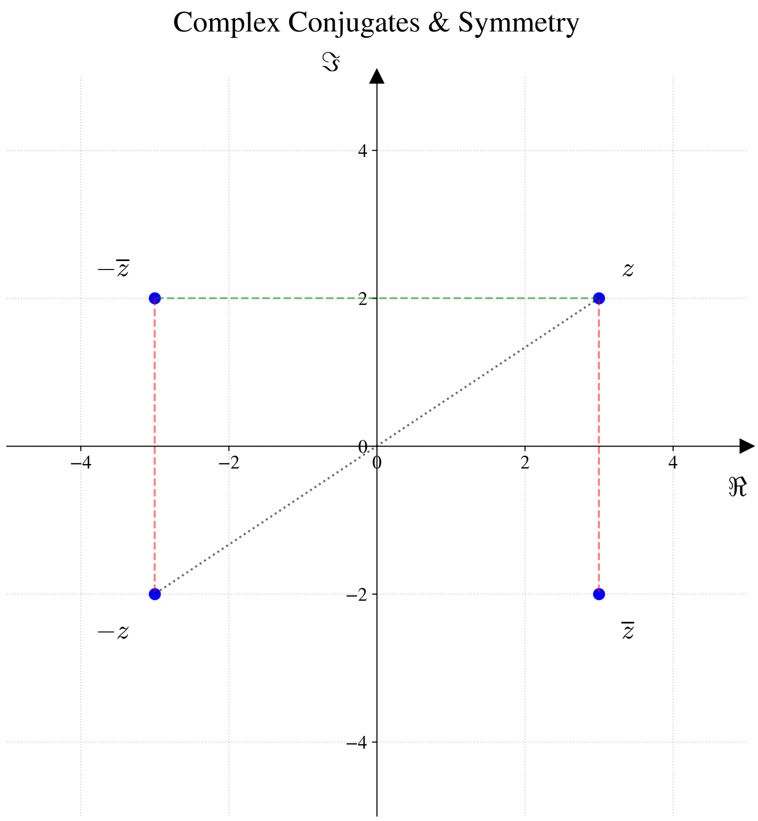

Figure 1.3: Geometric representation of the complex conjugate. The point $z = x + iy$ is reflected across the real axis to obtain $\bar{z} = x - iy$; the dashed line indicates the reflection. Analogously, $-z$ is the reflection across the origin, and $-\bar{z}$ is the reflection across the imaginary axis.

1.5 Complex Conjugates

Definition 1.5.1 (Conjugate)**.** The complex conjugate of a number $z = x + iy$ is defined as:

$$\bar{z} = x - iy$$

Geometrically, $\bar{z}$ is the reflection of the point z across the real axis.

Theorem 1.5.1 (Properties of Conjugates)**.** *For all $z, z_1, z_2 \in \mathbb{C}$:*

1. $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
2. $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$
3. $|z|^2 = z\bar{z}$
4. *Re*$(z) = \frac{z+\bar{z}}{2}$ *and Im*$(z) = \frac{z-\bar{z}}{2i}$

1.6 Exponential Form

Using polar coordinates (r, θ) where $x = r\cos\theta$ and $y = r\sin\theta$, we can express complex numbers in exponential form.

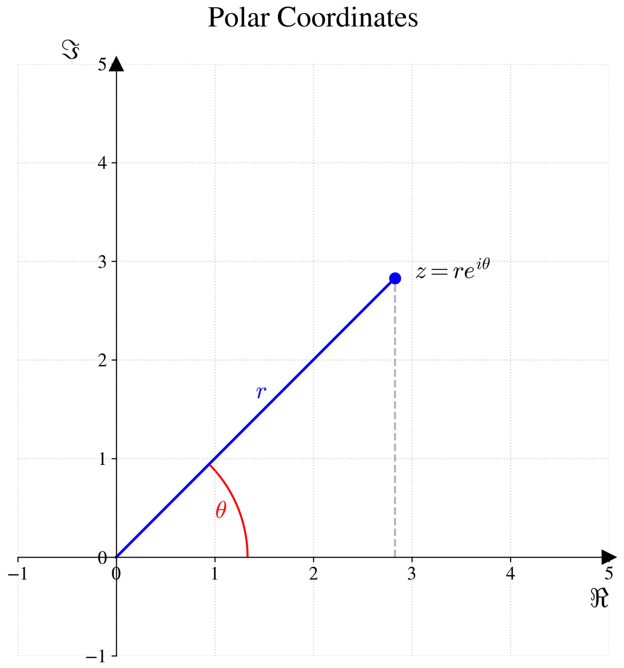

Figure 1.4: Exponential form of a complex number $z = re^{i\theta}$. The modulus r represents the distance from the origin, and the angle θ (argument) represents the direction from the positive real axis.

Definition 1.6.1 (Argument)**.** The argument of z, denoted $\arg(z)$, is the set of angles θ such that $z = r(\cos\theta + i\sin\theta)$.

$$\theta = \arg(z) = \Theta + 2n\pi, \quad n \in \mathbb{Z}$$

where Θ is the **Principal Argument**, denoted $\mathrm{Arg}(z)$, such that $-\pi < \Theta \leq \pi$.

Theorem 1.6.1 (Euler's Formula)**.** *For any real number θ,*

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Thus, any non-zero complex number can be written as:

$$z = re^{i\theta}$$

1.7 Products and Powers in Exponential Form

Exponential form makes multiplication and division significantly easier geometrically.

Theorem 1.7.1 (Multiplication in Polar Form)**.** *If $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$, then:*

$$z_1z_2 = r_1r_2e^{i(\theta_1+\theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$$

Remark 1.7.1. Multiplication by z scales the modulus by $|z|$ and rotates the vector by $\arg(z)$.

1.7.1 De Moivre's Formula

From the multiplication rule, we derive the formula for integer powers.

Theorem 1.7.2 (De Moivre's Formula)**.** *For any integer n,*

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

or equivalently, $(e^{i\theta})^n = e^{in\theta}$.

Example 1.7.1. Calculate $(1+i)^{10}$. First, convert $1+i$ to exponential form:

$$r = \sqrt{1^2+1^2} = \sqrt{2}, \quad \operatorname{Arg}(z) = \frac{\pi}{4}$$

$$z = \sqrt{2}e^{i\pi/4}$$

$$z^{10} = (\sqrt{2})^{10}e^{i10\pi/4} = 2^5 e^{i5\pi/2} = 32e^{i(\pi/2+2\pi)} = 32e^{i\pi/2} = 32i$$

1.8 Roots of Complex Numbers

Finding the n-th roots of a complex number z_0 involves solving $z^n = z_0$.

Theorem 1.8.1 (Roots)**.** *The n distinct n-th roots of $z_0 = r_0 e^{i\theta_0}$ are given by:*

$$z_k = \sqrt[n]{r_0}\exp\left(i\frac{\theta_0 + 2k\pi}{n}\right), \quad k = 0, 1, \ldots, n-1$$

Geometrically, the roots lie on a circle of radius $\sqrt[n]{r_0}$ centered at the origin and form the vertices of a regular n-sided polygon.

1.9 Regions in the Complex Plane

To do calculus, we need topological definitions regarding sets of points in the plane.

Definition 1.9.1 (Epsilon Neighborhood)**.** An ε-neighborhood of a point z_0 is the set of all points z such that:

$$|z - z_0| < \varepsilon$$

Definition 1.9.2 (Topological Concepts)**.** The following definitions apply to sets of complex numbers:

- **Interior Point:** A point z_0 in a set S is an interior point if there exists some neighborhood of z_0 completely contained in S.
- **Open Set:** A set S is open if every point in S is an interior point.
- **Closed Set:** A set S is closed if it contains all its boundary points.
- **Domain:** An open, connected set.
- **Region:** A domain together with some, none, or all of its boundary points.

Exercise 1.9.1. Determine if the set $S = \{z \in \mathbb{C} \mid |z| < 1\}$ is open or closed.

Solution: It is open, since for each point z_0 in S, we can find define $\varepsilon = \frac{1}{3}\operatorname{dist}(z_0, \partial S)$; then the set $\mathcal{U}_\varepsilon(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < \varepsilon\}$ is an ε-neighborhood that lies entirely within S.

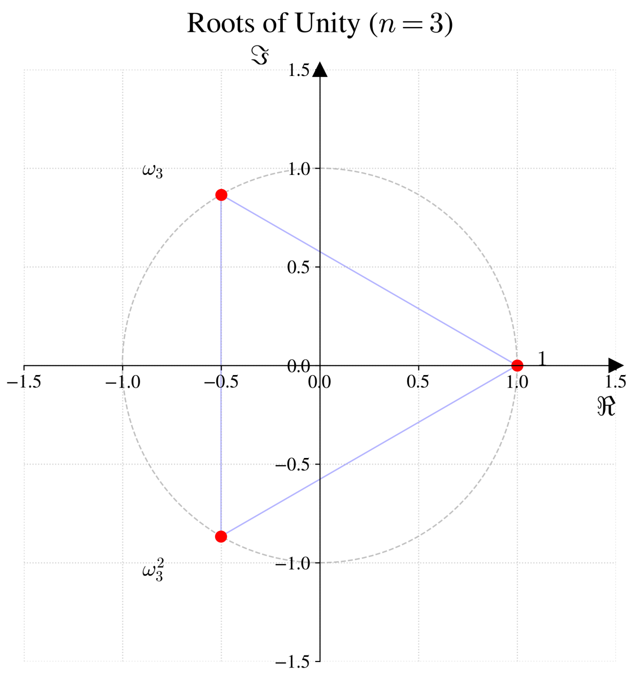

Figure 1.5: The n-th roots of a complex number z_0 are evenly spaced on a circle in the complex plane, forming the vertices of a regular n-sided polygon.

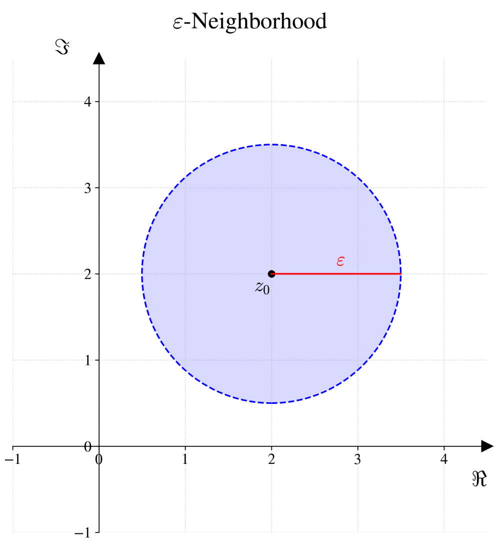

Figure 1.6: An ε-neighborhood around the point z_0. The shaded area represents all points z such that $|z - z_0| < \varepsilon$.

1.10 Exercises

Exercise 1.10.1. Write the following complex numbers in the form $x + iy$:

1. $(5 - 2i) + (3 + 4i)$
2. $(2 + 3i)(4 - i)$
3. $\frac{1}{1+i}$
4. $\frac{3-2i}{-1+i}$

Exercise 1.10.2. Find $\text{Re}(z)$ and $\text{Im}(z)$ for the following:

1. $z = \frac{1}{i}$
2. $z = (1 - i)^3$
3. $z = \frac{2+i}{3-4i} + \frac{2-i}{5i}$

Exercise 1.10.3. Let $z_1 = 1 - i$ and $z_2 = -2 + 4i$. Compute:

1. $z_1^2 - 2z_1 + 3$
2. $|2z_1 - 3z_2|$
3. $\text{Re}(z_1 z_2)$

Exercise 1.10.4. Verify that $(\sqrt{2} - i) - i(1 - \sqrt{2}i) = -2i$.

Exercise 1.10.5. Solve the following equation for z:

$$z(1 + i) = \bar{z} + (3 + 2i)$$

Hint: Let $z = x + iy$ and equate real and imaginary parts.

Exercise 1.10.6. Verify the following properties using $z = x + iy$:

1. $\overline{iz} = -i\bar{z}$
2. $\overline{\bar{z} + 3i} = z - 3i$
3. $|z|^2 = z\bar{z}$

Exercise 1.10.7. Prove that for any $z_1, z_2 \in \mathbb{C}$:

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$$

This is known as the *Parallelogram Law*.

Exercise 1.10.8. Show that $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\text{Re}(z_1\bar{z}_2)$.

Exercise 1.10.9. *(Hard)* If $|z| = 1$, prove that:

$$\left|\frac{z - w}{1 - \bar{w}z}\right| = 1$$

for any complex number w with $|w| \neq 1$.

Exercise 1.10.10. Write the following numbers in exponential form $re^{i\theta}$ $(-\pi < \theta \leq \pi)$:

1. $1 - i$
2. $-2\sqrt{3} - 2i$
3. $-i$
4. $(1 - i\sqrt{3})^2$

Exercise 1.10.11. Find the Principal Argument, $\text{Arg}(z)$, for:

1. $z = \frac{-2}{1+i\sqrt{3}}$
2. $z = (\sqrt{3} - i)^6$

Exercise 1.10.12. Using exponential form, show that:

$$(-1+i)^7 = -8(1+i)$$

Exercise 1.10.13. Compute the following powers and write the result in the form $x+iy$:

1. $(1+i)^{10}$
2. $\left(\frac{1-i\sqrt{3}}{2}\right)^{15}$
3. $(1-i)^{-8}$

Exercise 1.10.14. Compute the value of $(1 + \cos\frac{\pi}{3} + i\sin\frac{\pi}{3})^6$.
Hint: Use De Moivre's formula.

Exercise 1.10.15. Find all distinct roots of the following equations and locate them in the complex plane:

1. $z^3 = 1$
2. $z^2 + 2i = 0$
3. $z^4 = -16$

Exercise 1.10.16. Find the four values of $(-1)^{1/4}$.

Exercise 1.10.17. Solve the quadratic equation $z^2 + z + 1 = 0$ for z.

Exercise 1.10.18. Sketch the set of points determined by the condition:

1. $|z - 1 + i| = 1$
2. $|z + i| \leq 3$
3. $\text{Re}(\bar{z} - i) = 2$
4. $|2z - i| = 4$

Exercise 1.10.19. Describe the set of points z such that $\text{Im}(z^2) > 0$.

Exercise 1.10.20. Prove that if $1 + z + z^2 + \cdots + z^n = 0$, then z is a root of unity, i.e. $z = e^{\frac{2\pi ik}{n+1}}$ for some integer k.

Exercise 1.10.21. Establish the Lagrange trigonometric identity:

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin[(n+1/2)\theta]}{2\sin(\theta/2)}$$

where $0 < \theta < 2\pi$.

Chapter 2

Analytic Functions

2.1 Functions of a Complex Variable

A complex function f is a rule that assigns to each complex number z in a set $S \subseteq \mathbb{C}$ a complex number w. We write:

$$w = f(z)$$

Here, S is the **domain of definition** of f. Since $z = x + iy$, the function $f(z)$ can be decomposed into its real and imaginary parts:

$$f(z) = u(x, y) + iv(x, y)$$

where u and v are real-valued functions of the real variables x and y.

Figure Placeholder: Mapping

Caption: Draw two adjacent complex planes. Label the left one "z-plane" (xy-axes) and the right one "w-plane" (uv-axes). Draw a domain region S in the z-plane and a point z within it. Draw an arrow labeled "f" pointing to a region S' in the w-plane with a corresponding point w = f(z).

Example 2.1.1. Consider $f(z) = z^2$.

$$f(x + iy) = (x + iy)^2 = x^2 - y^2 + i(2xy)$$

Thus, $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$.

2.1.1 Example: The Joukowski Transformation

Let us examine the mapping defined by the function:

$$w = f(z) = z + \frac{1}{z}$$

This function is historically significant in aerodynamics for mapping circles into airfoil shapes. To understand the geometry of this mapping, it is most convenient to use polar coordinates $z = re^{i\theta}$. Substituting into the function:

$$w = re^{i\theta} + \frac{1}{re^{i\theta}} = r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$$

Separating the real and imaginary parts $w = u + iv$, we obtain:

$$u = \left(r + \frac{1}{r}\right)\cos\theta, \quad v = \left(r - \frac{1}{r}\right)\sin\theta \tag{2.1}$$

Mapping of the Unit Circle ($r = 1$)

Consider the unit circle $|z| = 1$. Here $r = 1$, so the equations become:

$$u = 2\cos\theta, \quad v = 0$$

As θ varies from 0 to 2π, u varies from 2 to -2 and back to 2. Thus, the unit circle in the z-plane is mapped onto the line segment $[-2, 2]$ on the real axis of the w-plane.

Figure Placeholder: Joukowski Unit Circle

Caption: Left: Draw the z-plane with a unit circle. Right: Draw the w-plane with a bold line segment on the real axis from -2 to 2. Draw arrows indicating the mapping.

Mapping of Circles ($r = c > 1$)

Now consider a circle $|z| = c$ where $c > 1$. The equations for u and v are:

$$u = A\cos\theta, \quad v = B\sin\theta$$

where $A = c + \frac{1}{c}$ and $B = c - \frac{1}{c}$. Using the identity $\cos^2\theta + \sin^2\theta = 1$, we can eliminate θ:

$$\frac{u^2}{A^2} + \frac{v^2}{B^2} = 1$$

This is the equation of an ellipse with semi-major axis A and semi-minor axis B. Note that the foci of this ellipse are located at $\pm\sqrt{A^2 - B^2} = \pm\sqrt{4} = \pm 2$, which are the endpoints of the slit created by the unit circle mapping.

Figure Placeholder: Joukowski Ellipse

Caption: Left: Draw the z-plane with concentric circles (e.g., r=1 and r=2). Right: Draw the w-plane showing the segment [-2,2] and an ellipse surrounding it. Show how the circle maps to the ellipse.

2.2 Limits and Continuity

The concept of a limit in the complex plane is similar to that in real calculus, but since the neighborhood is two-dimensional, z can approach z_0 from any direction.

Definition 2.2.1 (Limit)**.** Let f be defined in a neighborhood of z_0, except possibly at z_0 itself. We say that the limit of $f(z)$ as z approaches z_0 is w_0, written

$$\lim_{z \to z_0} f(z) = w_0$$

if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that:

$$|f(z) - w_0| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta$$

Figure Placeholder: Epsilon-Delta Limit

Caption: Draw a point z_0 in the z-plane with a small dashed circle of radius δ around it. Draw a point w_0 in the w-plane with a small dashed circle of radius ε around it. Show that any point z inside the δ-circle maps to a point f(z) inside the ε-circle.

Theorem 2.2.1 (Continuity)**.** *A function f is continuous at z_0 if:*

1. *$f(z_0)$ is defined,*
2. *$\lim_{z\to z_0} f(z)$ exists, and*
3. *$\lim_{z\to z_0} f(z) = f(z_0)$.*

2.3 Derivatives

Differentiation is the cornerstone of complex analysis. The definition is formally identical to the real case, but the implications are far more profound.

Definition 2.3.1 (Derivative)**.** Let f be defined in a neighborhood of z_0. The derivative of f at z_0, denoted $f'(z_0)$, is:

$$f'(z_0) = \lim_{\Delta z\to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

provided this limit exists.

If the limit exists, f is said to be **differentiable** at z_0. The approach $\Delta z \to 0$ can occur along the real axis ($\Delta x \to 0$), the imaginary axis ($i\Delta y \to 0$), or any other path. The limit must be independent of the path.

Remark 2.3.1. Standard differentiation rules (sum, product, quotient, chain rule) from calculus apply to complex derivatives.

2.3.1 Examples of Differentiability

To illustrate the definition, let us look at one function that is differentiable everywhere and one that is not differentiable anywhere.

Example 2.3.1 (Differentiable Function)**.** Consider the function $f(z) = z^2$. Using the definition of the derivative:

$$\begin{aligned} f'(z) &= \lim_{\Delta z\to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z} \\ &= \lim_{\Delta z\to 0} \frac{z^2 + 2z\Delta z + (\Delta z)^2 - z^2}{\Delta z} \\ &= \lim_{\Delta z\to 0} (2z + \Delta z) = 2z \end{aligned}$$

Since the limit $2z$ exists and is independent of the path taken by Δz, $f(z) = z^2$ is differentiable everywhere (it is an entire function).

Example 2.3.2 (Non-Differentiable Function)**.** Consider the function $f(z) = \overline{z} = x - iy$. We attempt to find the limit of the difference quotient:

$$\lim_{\Delta z\to 0} \frac{\overline{z + \Delta z} - \overline{z}}{\Delta z} = \lim_{\Delta z\to 0} \frac{\overline{\Delta z}}{\Delta z}$$

Let $\Delta z = \Delta x + i\Delta y$.

- **Path 1 (Along real axis):** Set $\Delta y = 0$, so $\Delta z = \Delta x$.

$$\lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

- **Path 2 (Along imaginary axis):** Set $\Delta x = 0$, so $\Delta z = i\Delta y$.

$$\lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = -1$$

Since the limits along these two paths are different ($1 \neq -1$), the derivative $f'(z)$ does not exist anywhere.

2.4 Cauchy-Riemann Equations

Since $f(z) = u(x, y) + iv(x, y)$, there is a necessary relationship between the partial derivatives of u and v for $f'(z)$ to exist.

Theorem 2.4.1 (Cauchy-Riemann Equations). *Suppose $f(z) = u(x, y) + iv(x, y)$ is differentiable at a point $z_0 = x_0 + iy_0$. Then the partial derivatives of u and v exist at (x_0, y_0) and satisfy the **Cauchy-Riemann equations**:*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Furthermore, the derivative is given by:

$$f'(z_0) = u_x + iv_x$$

Proof. Assume $f(z) = u(x, y) + iv(x, y)$ is differentiable at a point z. Then the limit

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

exists and is unique regardless of how Δz approaches 0.

Case 1: Approach along the horizontal axis ($\Delta y = 0$).
Let $\Delta z = \Delta x$. Then:

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x + \Delta x, y) - v(x, y)}{\Delta x}$$

$$f'(z) = u_x(x, y) + iv_x(x, y) \quad (*)$$

Case 2: Approach along the vertical axis ($\Delta x = 0$).
Let $\Delta z = i\Delta y$. Then:

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) - u(x, y)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x, y + \Delta y) - v(x, y)}{i\Delta y}$$

Using $1/i = -i$, we get:

$$f'(z) = -iu_y(x, y) + v_y(x, y) \quad (**)$$

Equating the real and imaginary parts of $(*)$ and $(**)$ gives the Cauchy-Riemann equations:

$$u_x = v_y \quad \text{and} \quad v_x = -u_y$$

□

Theorem 2.4.2 (Sufficient Conditions). *If the partial derivatives u_x, u_y, v_x, v_y exist, are continuous in a neighborhood of z_0, and satisfy the Cauchy-Riemann equations at z_0, then $f'(z_0)$ exists.*

2.4.1 Polar Form

Using $z = re^{i\theta}$, the Cauchy-Riemann equations in polar coordinates are:

$$ru_r = v_\theta, \quad u_\theta = -rv_r$$

and the derivative is given by:

$$f'(z) = e^{-i\theta}(u_r + iv_r)$$

Proof. Let $z = re^{i\theta}$ and $f(z) = u(r, \theta) + iv(r, \theta)$. Assuming f is differentiable at a non-zero point z, we can compute the partial derivatives with respect to r and θ using the chain rule.

Recall that $z = re^{i\theta}$.

1. Differentiate with respect to r:

$$\frac{\partial f}{\partial r} = f'(z)\frac{\partial z}{\partial r} = f'(z)e^{i\theta}$$

Also, directly from the components:

$$\frac{\partial f}{\partial r} = u_r + iv_r$$

Equating these implies:

$$f'(z) = e^{-i\theta}(u_r + iv_r) \quad (\dagger)$$

2. Differentiate with respect to θ:

$$\frac{\partial f}{\partial \theta} = f'(z)\frac{\partial z}{\partial \theta} = f'(z)(ire^{i\theta}) = irf'(z)e^{i\theta}$$

Also, directly from the components:

$$\frac{\partial f}{\partial \theta} = u_\theta + iv_\theta$$

Solving for $f'(z)$:

$$f'(z) = \frac{1}{ire^{i\theta}}(u_\theta + iv_\theta) = \frac{e^{-i\theta}}{r}(v_\theta - iu_\theta) \quad (\ddagger)$$

Equating (†) and (‡):

$$u_r + iv_r = \frac{1}{r}v_\theta - i\frac{1}{r}u_\theta$$

Matching real and imaginary parts yields the polar form:

$$u_r = \frac{1}{r}v_\theta, \quad v_r = -\frac{1}{r}u_\theta$$

□

2.5 Analytic Functions

Definition 2.5.1 (Analyticity)**.** A function f is **analytic** at a point z_0 if it is differentiable not only at z_0 but also at every point in some neighborhood of z_0. A function is analytic in a domain D if it is analytic at every point in D.

Definition 2.5.2 (Entire Function)**.** A function that is analytic at every point in the complex plane $\mathbb{C}$ is called an **entire function**. Examples include polynomials and e^z.

Theorem 2.5.1. *If $f'(z) = 0$ everywhere in a domain D, then $f(z)$ is constant in D.*

2.6 Harmonic Functions

Analytic functions are closely related to the Laplace equation from physics and engineering.

Definition 2.6.1 (Harmonic Function)**.** A real-valued function $H(x, y)$ is **harmonic** in a domain if it has continuous partial derivatives of the first and second order and satisfies Laplace's equation:

$$H_{xx}(x, y) + H_{yy}(x, y) = 0$$

Theorem 2.6.1. *If $f(z) = u(x, y)+iv(x, y)$ is analytic in a domain D, then its component functions u and v are harmonic in D.*

Definition 2.6.2 (Harmonic Conjugate)**.** If u and v are harmonic in a domain D and their first-order partial derivatives satisfy the Cauchy-Riemann equations, then v is called a **harmonic conjugate** of u.

Figure Placeholder: Level Curves

Caption: Draw the xy-plane. Sketch a set of curves representing $u(x, y) = c_1$ (solid lines). Sketch a second set of curves representing $v(x, y) = c_2$ (dashed lines) that intersect the solid lines at 90 degree angles (orthogonal trajectories), illustrating the property of harmonic conjugates.

2.7 Exercises

Exercise 2.7.1. Determine the domain of definition for the following functions:

1. $f(z) = \frac{1}{z^2+1}$
2. $f(z) = \text{Arg}\left(\frac{1}{z}\right)$

Exercise 2.7.2. Using the definition of the derivative (limits), show that the function $f(z) = \text{Re}(z)$ is nowhere differentiable.

Exercise 2.7.3. Use the Cauchy-Riemann equations to determine where the following functions are differentiable and where they are analytic:

1. $f(z) = \bar{z}$

2. $f(z) = e^{-x}(\cos y - i \sin y)$
3. $f(z) = z^2 + z$

Exercise 2.7.4. Show that the function $u(x, y) = 2x(1-y)$ is harmonic. Find a function $v(x, y)$ such that $f(z) = u + iv$ is analytic. Express $f(z)$ in terms of z.

Exercise 2.7.5. Prove that if $f(z)$ is analytic in a domain D and $|f(z)|$ is constant in D, then $f(z)$ itself must be constant in D.

Exercise 2.7.6. Using the polar form of the Cauchy-Riemann equations, show that the function $f(z) = \sqrt{r}e^{i\theta/2}$ (where $r > 0$ and $-\pi < \theta < \pi$) is analytic in its domain. Compute $f'(z)$.

Exercise 2.7.7. Consider the function $f(z) = |z|^2$.

1. Write $f(z)$ in terms of x and y (i.e., find $u(x, y)$ and $v(x, y)$).
2. Apply the Cauchy-Riemann equations to determine the set of points where $f'(z)$ exists.
3. Is $f(z)$ analytic at any point? Explain the difference between your answer here and the result in part 2.