Midterm Exam 2

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Name: ______________________________ **Version:** ☐ UG ☐ G

Instructions

- **Undergraduate:** Complete Exercises 1–4. **Graduate:** Complete Exercises 1–5.

Reference: Definitions & Theorems

Principal Logarithm: $\text{Log}(z) = \ln|z| + i\text{Arg}(z)$, where $-\pi < \text{Arg}(z) \leq \pi$.

Cauchy-Goursat Theorem: If f is analytic everywhere inside and on a simple closed contour C, then $\int_C f(z)dz = 0$.

Cauchy Integral Formula: If f is analytic inside and on a simple closed contour C, and z_0 is any point interior to C, then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz$$

Exercise 1

(a) Using the principal branch of the logarithm, compute the principal value of $(-i)^i$. Express your final answer in the form $x + iy$.

(b) Find the exact location of the branch cut for the function $f(z) = \text{Log}(z - 2 + i)$. Sketch the branch cut in the complex plane below.

Exercise 2

Evaluate the following contour integral directly by parameterizing the contour:

$$\int_C \bar{z}\, dz$$

where C is the upper half of the circle $|z| = 3$, traversed counterclockwise from $z = 3$ to $z = -3$.

Exercise 3

Evaluate the following integrals over the given simple closed contours, all traversed counterclockwise. Justify your reasoning.

(a) $\displaystyle\int_{|z|=1} \frac{e^z}{z-3}\, dz$

(b) $\displaystyle\int_{|z|=2} \frac{1}{z(z-1)}\, dz$

(Hint: You may use partial fractions or split the contour into two disjoint loops.)

Exercise 4

Use the generalized Cauchy Integral Formula to evaluate the following integral:

$$\int_{|z|=2} \frac{z^3 + 2z}{(z-i)^3}\, dz$$

where the circle $|z| = 2$ is traversed in the counterclockwise direction.

Exercise 5 (Graduate Only)

Consider the function:

$$f(z) = \mathrm{Log}(z+2)$$

for $x < -2$ compute the following limits:

$$\lim_{y\to 0^+} f(x+iy) \quad \text{and} \quad \lim_{y\to 0^+} f(x-iy)\,,$$

deduce that

$$\lim_{y\to 0^+} f(x+iy) - f(x-iy) = 2\pi i\,.$$