Midterm Exam 3 (Retake)

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Name: ______________________ **Version:** ☐ UG ☐ G

Instructions

- **Undergraduate:** Complete Exercises 1–4. **Graduate:** Complete Exercises 1–5.
- Show all your work. Explicitly state any contours used, justify limits as $R \to \infty$, and clearly indicate the locations and orders of your poles.

Reference: Definitions & Theorems

Cauchy's Residue Theorem: If f is analytic inside and on a simple closed contour C, except for isolated singularities z_k inside C, then $\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f, z_k)$.

Residue at a Simple Pole: If f has a simple pole at z_0, then $\text{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$. Alternatively, if $f(z) = P(z)/Q(z)$ where $P(z_0) \neq 0$, $Q(z_0) = 0$, and $Q'(z_0) \neq 0$, then $\text{Res}(f, z_0) = \dfrac{P(z_0)}{Q'(z_0)}$.

Residue at a Pole of Order m: If f has a pole of order m at z_0, then

$$\text{Res}(f, z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z)\right]$$

Exercise 1

Consider the rational function:

$$f(z) = \frac{3}{(z-1)(z-4)}$$

(a) Find the partial fraction decomposition of $f(z)$.

(b) Compute the Laurent series expansion for $f(z)$ centered at $z_0 = 0$ in the three regions:

i. The disk $|z| < 1$.

ii. The annulus $1 < |z| < 4$.

iii. The exterior region $|z| > 4$.

Exercise 2

Evaluate the following contour integral, where C is the circle $|z| = 3$ oriented counterclockwise:

$$\oint_C \frac{z^2 - 3z}{(z + 2i)^2(z - 1)}\, dz$$

Exercise 3

Use contour integration to evaluate the following improper real integral:

$$\int_{-\infty}^{\infty} \frac{1}{x^4 + 3^4}\, dx$$

Exercise 4

Use the Residue Theorem to evaluate the following definite integral:

$$\int_0^{2\pi} \cos^4(\theta)\, d\theta$$

Exercise 5 (Graduate Only)

Use the Residue Theorem to evaluate the infinite series:

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + 9}$$

(Hint: Integrate the function $f(z) = \frac{\pi \cot(\pi z)}{z^2+9}$ over a large square contour C_N. You may assume that the integral over C_N vanishes as $N \to \infty$.)