Practice Midterm Exam 1 Solutions

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Exercise 1: Roots and Polynomials

Consider the equation $z^5 = -32$.

(a) **Find all complex solutions in exponential form.**

Solution:

Write -32 in exponential form: $r = 32$ and $\theta = \pi$.

$$-32 = 32e^{i\pi}$$

The roots are given by $z_k = \sqrt[5]{32}\exp\left(i\frac{\pi+2\pi k}{5}\right)$ for $k = 0, 1, 2, 3, 4$. Since $\sqrt[5]{32} = 2$, the roots are:

$$\begin{aligned} z_0 &= 2e^{i\pi/5} \\ z_1 &= 2e^{i3\pi/5} \\ z_2 &= 2e^{i5\pi/5} = 2e^{i\pi} = -2 \\ z_3 &= 2e^{i7\pi/5} \\ z_4 &= 2e^{i9\pi/5} \end{aligned}$$

(b) **Sketch the roots in the complex plane.**

Solution:

The roots lie on a circle of radius 2. They form a regular pentagon.

- $z_2 = -2$ is on the negative real axis.
- z_0 and z_4 are in Q1 and Q4 respectively (conjugates).
- z_1 and z_3 are in Q2 and Q3 respectively (conjugates).

(c) **Factor $P(z) = z^5 + 32$ into polynomials with real coefficients.**

Solution:

The roots come in conjugate pairs, plus one real root.

1. **Real factor:** From $z_2 = -2$, we have the factor $(z + 2)$.

2. **Pair 1 (z_0, z_4):** $z_0 = 2e^{i\pi/5}$ and $z_4 = \overline{z_0} = 2e^{-i\pi/5}$.

$$(z - z_0)(z - \overline{z_0}) = z^2 - 2\text{Re}(z_0)z + |z_0|^2$$
$$= z^2 - 4\cos(\pi/5)z + 4$$

3. **Pair 2 (z_1, z_3):** $z_1 = 2e^{i3\pi/5}$ and $z_3 = \overline{z_1} = 2e^{-i3\pi/5}$.

$$(z - z_1)(z - \overline{z_1}) = z^2 - 2\text{Re}(z_1)z + |z_1|^2$$
$$= z^2 - 4\cos(3\pi/5)z + 4$$

Final Factorization:

$$P(z) = (z+2)(z^2 - 4\cos(\frac{\pi}{5})z + 4)(z^2 - 4\cos(\frac{3\pi}{5})z + 4)$$

(Optional: exact values $\cos(\frac{\pi}{5}) = \frac{1+\sqrt{5}}{4}$ *and* $\cos(\frac{3\pi}{5}) = \frac{1-\sqrt{5}}{4}$ *can be substituted.)*

Exercise 2: Limits

Compute the limits. If one does not exist, show why using two paths.

(a) $\displaystyle\lim_{z\to 1+i} \frac{z^2-2z+2}{z-(1+i)}$

Solution:

First, check if the numerator is zero at $z_0 = 1+i$:

$$(1+i)^2 - 2(1+i) + 2 = (1+2i-1) - 2 - 2i + 2 = 2i - 2i = 0$$

Since we have a $0/0$ form, we can factor the numerator. The roots of $z^2-2z+2=0$ are found by the quadratic formula:

$$z = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

Thus, the numerator factors as $(z-(1+i))(z-(1-i))$.

$$\begin{aligned}\lim_{z\to 1+i} \frac{(z-(1+i))(z-(1-i))}{z-(1+i)} &= \lim_{z\to 1+i}(z-(1-i)) \\ &= (1+i)-(1-i) \\ &= \mathbf{2i}\end{aligned}$$

Alternatively, using L'Hôpital's Rule (differentiation wrt z): $\lim \frac{2z-2}{1} = 2(1+i)-2 = 2i$.

(b) $\displaystyle\lim_{z\to 0} \left(\frac{z}{\bar{z}}\right)^2$

Solution:

We test different paths approaching 0.

Path 1: Along the real axis ($y=0$). Here $z=x$, so $\bar{z}=x$.

$$\lim_{x\to 0}\left(\frac{x}{x}\right)^2 = 1^2 = 1$$

Path 2: Along the line $y=x$ ($\theta=\pi/4$). Here $z = re^{i\pi/4}$ and $\bar{z} = re^{-i\pi/4}$.

$$\left(\frac{re^{i\pi/4}}{re^{-i\pi/4}}\right)^2 = \left(e^{i(\pi/4-(-\pi/4))}\right)^2 = (e^{i\pi/2})^2 = (i)^2 = -1$$

Since the limits along two different paths are distinct ($1 \neq -1$), **the limit does not exist.**

Exercise 3: Powers

Let $z = -\sqrt{3} + i$.

(a) **Write z in exponential form.**

Solution:

1. Modulus: $r = \sqrt{(-\sqrt{3})^2 + (1)^2} = \sqrt{3+1} = \sqrt{4} = 2$.

2. Argument: Since $x < 0$ and $y > 0$, z is in Quadrant II. The reference angle is $\alpha = \tan^{-1}(|1/-\sqrt{3}|) = \tan^{-1}(1/\sqrt{3}) = \pi/6$.

$$\text{Arg}(z) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

$$z = 2e^{i5\pi/6}$$

(b) **Compute z^{12} and express the answer in rectangular form.**

Solution:

Using De Moivre's property:

$$\begin{aligned} z^{12} &= (2e^{i5\pi/6})^{12} \\ &= 2^{12} \cdot e^{i12(5\pi/6)} \\ &= 4096 \cdot e^{i10\pi} \end{aligned}$$

Since 10π is an integer multiple of 2π, $e^{i10\pi} = \cos(10\pi) + i\sin(10\pi) = 1$.

$$z^{12} = \mathbf{4096}$$

(Rectangular form: $4096 + 0i$).

Exercise 4: Harmonic Conjugates

Consider $u(x, y) = x^3 - 3xy^2$.

(a) **Verify that $u(x, y)$ is harmonic.**

Solution:

We compute the second partial derivatives:

$$u_x = 3x^2 - 3y^2 \implies u_{xx} = 6x$$
$$u_y = -6xy \implies u_{yy} = -6x$$

Summing them:

$$\Delta u = u_{xx} + u_{yy} = 6x + (-6x) = 0$$

Thus, u is harmonic.

(b) **Find the harmonic conjugate $v(x, y)$.**

Solution:

We use the Cauchy-Riemann equations: $u_x = v_y$ and $u_y = -v_x$.

1. Integrate $v_y = u_x$ with respect to y:

$$v_y = 3x^2 - 3y^2$$
$$v = \int (3x^2 - 3y^2)dy = 3x^2y - y^3 + h(x)$$

2. Differentiate with respect to x and use $v_x = -u_y$:

$$v_x = 6xy + h'(x)$$

From u, we know $-u_y = -(-6xy) = 6xy$.

$$6xy + h'(x) = 6xy \implies h'(x) = 0 \implies h(x) = C$$

Taking $C = 0$, we get:

$$v(x, y) = 3x^2y - y^3$$

(c) **Bonus: Write $f(z)$ explicitly in terms of z.**

Solution:

$$f(z) = u + iv = (x^3 - 3xy^2) + i(3x^2y - y^3)$$

Recall that $(x + iy)^3 = x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3 = x^3 + i3x^2y - 3xy^2 - iy^3$.
Grouping terms:

$$(x + iy)^3 = (x^3 - 3xy^2) + i(3x^2y - y^3)$$

Therefore:

$$f(z) = z^3$$

Exercise 5: Theory (Graduate Only)

Proposition: *If $f(z)$ is analytic in a domain D and $|f(z)| = c$ (constant), then $f(z)$ is constant.*

Solution:

Let $f(z) = u + iv$. Then $|f(z)|^2 = u^2 + v^2 = c^2$. Differentiate this equation with respect to x and y:

$$2uu_x + 2vv_x = 0 \implies uu_x + vv_x = 0 \tag{1}$$
$$2uu_y + 2vv_y = 0 \implies uu_y + vv_y = 0 \tag{2}$$

Using the Cauchy-Riemann equations ($u_y = -v_x, v_y = u_x$), substitute into (2):

$$u(-v_x) + v(u_x) = 0 \implies -uv_x + vu_x = 0 \tag{3}$$

Now we have a system for unknowns u_x and v_x using (1) and (3):

$$\begin{cases} uu_x + vv_x = 0 \\ vu_x - uv_x = 0 \end{cases}$$

Multiply the first by u and the second by v, then add:

$$(u^2u_x + uvv_x) + (v^2u_x - uvv_x) = 0$$

$$(u^2 + v^2)u_x = 0$$

Since $u^2 + v^2 = c^2$, we have $c^2u_x = 0$.

- If $c = 0$, then $u = 0, v = 0 \implies f = 0$ (constant).
- If $c \neq 0$, then $u_x = 0$. Similarly, we can show $v_x = 0$, $u_y = 0$, $v_y = 0$.

Since all partial derivatives are zero in the domain D, u and v are constant, so $f(z)$ is constant.