Practice Midterm Exam 2

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Name: ______________________________ **Version:** ☐ UG ☐ G

Instructions

- **Undergraduate:** Complete Exercises 1–4. **Graduate:** Complete Exercises 1–5.
- You have 50 minutes to complete the exam.
- Show all work clearly. Justify when you are using a specific theorem.

Reference: Definitions & Theorems

Principal Logarithm: $\text{Log}(z) = \ln|z| + i\text{Arg}(z)$, where $-\pi < \text{Arg}(z) \leq \pi$.

Cauchy-Goursat Theorem: If f is analytic everywhere inside and on a simple closed contour C, then $\int_C f(z)dz = 0$.

Cauchy Integral Formula: If f is analytic inside and on a simple closed contour C, and z_0 is any point interior to C, then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz$$

Exercise 1

(a) Using the principal branch of the logarithm, compute the principal value of $(1-i)^{1+i}$. Express your final answer in the algebraic form $x + iy$.

(b) Find the exact location of the branch cut for the function $f(z) = \text{Log}(z^2+4)$. Sketch the branch cut in the complex plane below.

Exercise 2

Evaluate the following contour integral directly by parameterizing the contour:

$$\oint_C \frac{z}{\bar{z}}\, dz$$

where C is the boundary of the upper half-disk $|z| \leq 2$ with $\text{Im}(z) \geq 0$, traversed in the counterclockwise direction.
(Hint: You must split the integral into two separate paths: the semicircular arc and the straight line segment on the real axis.)

Exercise 3

Evaluate the following integrals over the given simple closed contours, all traversed counterclockwise. Justify your reasoning explicitly.

(a) $\displaystyle\int_{|z-2|=1} \frac{\mathrm{Log}(z+3)}{z^2+16}\,dz$

(b) $\displaystyle\int_{|z|=3} \frac{\cos(\pi z)}{z^2-1}\,dz$

(Hint: For part (b), you may use partial fractions or split the contour into two disjoint loops.)

Exercise 4

Use the generalized Cauchy Integral Formula to evaluate the following integral:

$$\int_{|z-i|=1} \frac{z\, e^{i\pi z}}{(z^2+1)^2}\, dz$$

where the circle $|z-i|=1$ is traversed in the counterclockwise direction.
(Hint: Be careful to isolate the part of the integrand that is analytic inside the given contour before differentiating.)

Exercise 5 (Graduate Only)

Consider the function defined by the product of two principal square roots:

$$h(z) = (z-1)^{\frac{1}{2}}(z+1)^{\frac{1}{2}} = \exp\left(\frac{1}{2}\text{Log}(z-1)\right)\exp\left(\frac{1}{2}\text{Log}(z+1)\right)$$

For the interval $x < -1$ on the negative real axis, compute the following limits as we approach the real axis from the upper half-plane and the lower half-plane:

$$\lim_{y\to 0^+} h(x+iy) \quad \text{and} \quad \lim_{y\to 0^+} h(x-iy)\,.$$

Deduce that:

$$\lim_{y\to 0^+} h(x+iy) = \lim_{y\to 0^+} h(x-iy)\,.$$