Practice Midterm Exam 2 Solutions

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Exercise 1

(a) Using the principal branch of the logarithm, compute the principal value of $(1-i)^{1+i}$. Express your final answer in the algebraic form $x+iy$.

Solution: By definition, $(1-i)^{1+i} = \exp((1+i)\text{Log}(1-i))$. First, evaluate the principal logarithm:

$$\text{Log}(1-i) = \ln|1-i| + i\text{Arg}(1-i) = \ln\sqrt{2} - i\frac{\pi}{4}$$

Next, multiply by the exponent:

$$(1+i)\left(\ln\sqrt{2} - i\frac{\pi}{4}\right) = \ln\sqrt{2} - i\frac{\pi}{4} + i\ln\sqrt{2} - i^2\frac{\pi}{4}$$

Group the real and imaginary parts:

$$= \left(\ln\sqrt{2} + \frac{\pi}{4}\right) + i\left(\ln\sqrt{2} - \frac{\pi}{4}\right)$$

Exponentiate this result to get the final algebraic form:

$$\exp\left(\ln\sqrt{2} + \frac{\pi}{4}\right)\left[\cos\left(\ln\sqrt{2} - \frac{\pi}{4}\right) + i\sin\left(\ln\sqrt{2} - \frac{\pi}{4}\right)\right]$$

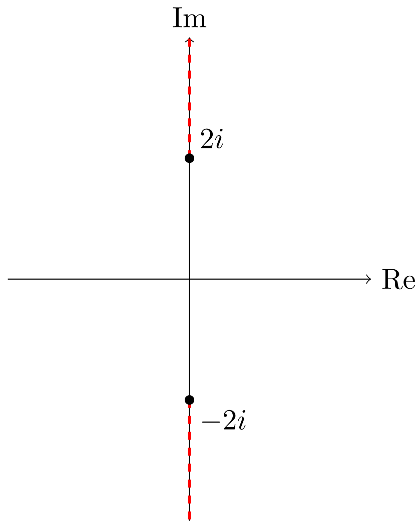
$$= \sqrt{2}e^{\pi/4}\cos\left(\frac{\ln 2}{2} - \frac{\pi}{4}\right) + i\sqrt{2}e^{\pi/4}\sin\left(\frac{\ln 2}{2} - \frac{\pi}{4}\right)$$

(b) Find the exact location of the branch cut for the function $f(z) = \text{Log}(z^2+4)$. Sketch the branch cut in the complex plane below.

Solution: The branch cut occurs where $z^2 + 4 \leq 0$. Let $z = x + iy$. Then $z^2 + 4 = (x^2 - y^2 + 4) + 2ixy$. For this to be purely real, the imaginary part must be zero: $2xy = 0 \implies x = 0$ or $y = 0$.

- If $y = 0$, the real part is x^2+4. Setting this ≤ 0 gives $x^2 \leq -4$, which has no real solutions.
- If $x = 0$, the real part is $-y^2+4$. Setting this ≤ 0 gives $y^2 \geq 4 \implies y \geq 2$ or $y \leq -2$.

Thus, the branch cut consists of two vertical rays on the imaginary axis: from $2i$ upwards to $+i\infty$, and from $-2i$ downwards to $-i\infty$.


Im
2i
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Exercise 2

Evaluate the following contour integral directly by parameterizing the contour:

$$\oint_C \frac{z}{\bar{z}}\,dz$$

where C is the boundary of the upper half-disk $|z| \leq 2$ with $\text{Im}(z) \geq 0$, traversed in the counterclockwise direction.

Solution: The contour C consists of two parts: the straight line segment C_1 along the real axis from -2 to 2, and the upper semicircular arc C_2 from 2 to -2.

Path 1 (C_1): Let $z(x) = x$ for $x \in [-2, 2]$. $dz = dx$ and $\bar{z} = x$. (Note: At $x = 0$, the integrand is technically undefined, but bounded, so the improper integral converges).

$$\int_{C_1} \frac{z}{\bar{z}}\,dz = \int_{-2}^{2} \frac{x}{x}\,dx = \int_{-2}^{2} 1\,dx = 4$$

Path 2 (C_2): Let $z(t) = 2e^{it}$ for t from 0 to π. $dz = 2ie^{it}\,dt$ and $\bar{z} = 2e^{-it}$.

$$\int_{C_2} \frac{z}{\bar{z}}\,dz = \int_0^{\pi} \frac{2e^{it}}{2e^{-it}}(2ie^{it})\,dt = \int_0^{\pi} e^{2it}(2ie^{it})\,dt = \int_0^{\pi} 2ie^{3it}\,dt$$

$$= 2i\left[\frac{e^{3it}}{3i}\right]_0^{\pi} = \frac{2}{3}(e^{3\pi i} - e^0) = \frac{2}{3}(-1-1) = -\frac{4}{3}$$

Total Integral: Sum the two paths:

$$\oint_C \frac{z}{\bar{z}}\,dz = 4 - \frac{4}{3} = \frac{8}{3}$$

Exercise 3

Evaluate the following integrals over the given simple closed contours, all traversed counterclockwise. Justify your reasoning explicitly.

(a) $\displaystyle\int_{|z-2|=1} \frac{\text{Log}(z+3)}{z^2+16}\, dz$

Solution: We first locate the singularities of the integrand.

(a) The denominator $z^2 + 16 = 0$ gives a singularity at $z = \pm 4i$.

(b) The principal logarithm $\text{Log}(z+3)$ has a branch where $z + 3 \leq 0 \implies x \leq -3$.

The contour $|z-2| = 1$ is a circle centered at $z = 2$ with radius 1. The points inside this contour have real parts strictly between 1 and 3. The poles $\pm 4i$ are outside the contour. The branch cut $x \leq -3$ is far to the left of the contour. Because the integrand is completely analytic inside and on the contour, by the **Cauchy-Goursat Theorem**, the integral is 0.

(b) $\displaystyle\int_{|z|=3} \frac{\cos(\pi z)}{z^2-1}\, dz$

Solution: The denominator factors as $(z-1)(z+1)$, giving singularities at $z = 1$ and $z = -1$. Both lie inside the circle $|z| = 3$. We use partial fraction decomposition:

$$\frac{1}{(z-1)(z+1)} = \frac{1/2}{z-1} - \frac{1/2}{z+1}$$

Split the integral and apply the Cauchy Integral Formula to each part with $f(z) = \cos(\pi z)$:

$$\frac{1}{2}\int_{|z|=3} \frac{\cos(\pi z)}{z-1}\, dz - \frac{1}{2}\int_{|z|=3} \frac{\cos(\pi z)}{z+1}\, dz$$

$$= \frac{1}{2}(2\pi i f(1)) - \frac{1}{2}(2\pi i f(-1))$$

Evaluating f: $f(1) = \cos(\pi) = -1$ and $f(-1) = \cos(-\pi) = -1$.

$$= \frac{1}{2}(-2\pi i) - \frac{1}{2}(-2\pi i) = -\pi i + \pi i = 0$$

Exercise 4

Use the generalized Cauchy Integral Formula to evaluate the following integral:

$$\int_{|z-i|=1} \frac{z\, e^{i\pi z}}{(z^2+1)^2}\, dz$$

Solution: Factor the denominator: $(z^2+1)^2 = (z-i)^2(z+i)^2$. The contour $|z-i| = 1$ encloses only the singularity at $z = i$. The singularity at $z = -i$ is outside. We isolate the analytic part of the integrand:

$$\int_C \frac{\left[\frac{ze^{i\pi z}}{(z+i)^2}\right]}{(z-i)^2}\, dz$$

Let $f(z) = \frac{ze^{i\pi z}}{(z+i)^2}$. By the Extended CIF ($n = 1$), the integral is $2\pi i f'(i)$. Compute $f'(z)$ using the quotient rule:

$$f'(z) = \frac{(e^{i\pi z} + i\pi z e^{i\pi z})(z+i)^2 - (ze^{i\pi z})(2)(z+i)}{(z+i)^4}$$

Evaluate at $z = i$. Note that $e^{i\pi(i)} = e^{-\pi}$, and $(i+i) = 2i$:

$$f'(i) = \frac{(e^{-\pi} - \pi e^{-\pi})(-4) - (ie^{-\pi})(2)(2i)}{16}$$

$$= \frac{-4e^{-\pi}(1-\pi) - (-4e^{-\pi})}{16} = \frac{-4e^{-\pi} + 4\pi e^{-\pi} + 4e^{-\pi}}{16} = \frac{4\pi e^{-\pi}}{16} = \frac{\pi e^{-\pi}}{4}$$

Multiply by $2\pi i$:

$$\text{Result} = 2\pi i \left(\frac{\pi e^{-\pi}}{4}\right) = \frac{i\pi^2}{2e^{\pi}}$$

Exercise 5 (Graduate Only)

Consider the function $h(z) = \exp\left(\frac{1}{2}\text{Log}(z-1)\right)\exp\left(\frac{1}{2}\text{Log}(z+1)\right)$. Compute the limits from above and below the real axis for $x < -1$ and deduce they are equal.

Solution: Let $x < -1$.

Limit from above ($y \to 0^+$): As z approaches the real axis from above, $z-1$ approaches $x-1$ (a negative number) from the upper half-plane, giving an argument of π. The first term approaches $\exp(\frac{1}{2}(\ln|x-1|+i\pi)) = \sqrt{|x-1|}e^{i\pi/2} = i\sqrt{|x-1|}$. Similarly, $z+1$ approaches a negative number with argument π. The second term approaches $i\sqrt{|x+1|}$. The product limit is: $(i)(i)\sqrt{|x-1||x+1|} = -\sqrt{x^2-1}$.

Limit from below ($y \to 0^-$): As z approaches from below, $z-1$ and $z+1$ approach negative numbers from the lower half-plane, giving arguments of $-\pi$. The first term approaches $\exp(\frac{1}{2}(\ln|x-1|-i\pi)) = -i\sqrt{|x-1|}$. The second term approaches $-i\sqrt{|x+1|}$. The product limit is: $(-i)(-i)\sqrt{|x-1||x+1|} = -\sqrt{x^2-1}$.

Deduction: Because the limits from above and below are both exactly $-\sqrt{x^2-1}$, the function has no jump discontinuity on the ray $(-\infty, -1)$. The overlapping branch cuts of the two individual square roots cancel each other out!