

# Midterm Exam Solutions

Instructor: Guido Mazzuca

Name: \_\_\_\_\_

Version:  UG  G

## Instructions

- **Undergraduate:** Complete Exercises 1–4.    **Graduate:** Complete Exercises 1–5.
- Show all work clearly.

## Definitions

**Cauchy-Riemann Equations:** If  $f = u + iv$  is differentiable, then  $u_x = v_y$  and  $u_y = -v_x$ . **Harmonic Conjugate:**  $v$  is the harmonic conjugate of  $u$  if  $u_{xx} + u_{yy} = 0$ ,  $v_{xx} + v_{yy} = 0$ , and they satisfy C-R equations.

## Exercise 1

Consider the equation  $z^3 = 8$ .

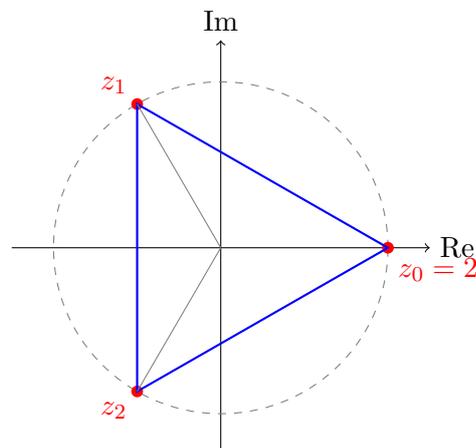
- (a) Find all complex solutions in exponential form.

**Solution:**

We rewrite 8 in polar form:  $8 = 8e^{i(0+2k\pi)}$ . The roots are given by  $z_k = \sqrt[3]{8}e^{i\frac{0+2k\pi}{3}}$  for  $k = 0, 1, 2$ .

- $k = 0$ :  $z_0 = 2e^{i0} = 2$
- $k = 1$ :  $z_1 = 2e^{i2\pi/3}$
- $k = 2$ :  $z_2 = 2e^{i4\pi/3}$

- (b) Sketch the roots in the complex plane.



(c) Factor  $P(z) = z^3 - 8$  into polynomials with **real coefficients**.

**Solution:**

We know that  $z^3 - 8 = (z - z_0)(z - z_1)(z - z_2)$ . The first factor is real:  $(z - 2)$ . The other two roots are complex conjugates:  $z_1 = -1 + i\sqrt{3}$  and  $z_2 = -1 - i\sqrt{3}$ . Multiplying the conjugate factors:

$$\begin{aligned}(z - z_1)(z - \bar{z}_1) &= z^2 - 2\operatorname{Re}(z_1)z + |z_1|^2 \\ &= z^2 - 2(-1)z + 2^2 = z^2 + 2z + 4\end{aligned}$$

Thus, the factorization is:

$$P(z) = (z - 2)(z^2 + 2z + 4)$$

## Exercise 2

Compute the limits. If one does not exist, show why using two paths.

(a)  $\lim_{z \rightarrow 2i} \frac{z^2 + 4}{z - 2i}$

**Solution:**

Direct substitution yields  $0/0$ . We factor the numerator using  $z^2 + 4 = z^2 - (2i)^2 = (z - 2i)(z + 2i)$ .

$$\begin{aligned} \lim_{z \rightarrow 2i} \frac{(z - 2i)(z + 2i)}{z - 2i} &= \lim_{z \rightarrow 2i} (z + 2i) \\ &= 2i + 2i = 4i \end{aligned}$$

(b)  $\lim_{z \rightarrow 0} \frac{\operatorname{Im}(z)}{z}$

**Solution:**

Let  $z = x + iy$ . Then  $\operatorname{Im}(z) = y$ . The limit is  $\lim_{(x,y) \rightarrow (0,0)} \frac{y}{x+iy}$ .

**Path 1: Along the real axis ( $y = 0$ ).**

$$\lim_{x \rightarrow 0} \frac{0}{x + 0} = 0$$

**Path 2: Along the imaginary axis ( $x = 0$ ).**

$$\lim_{y \rightarrow 0} \frac{y}{0 + iy} = \lim_{y \rightarrow 0} \frac{y}{iy} = \frac{1}{i} = -i$$

Since the limits along two different paths are distinct ( $0 \neq -i$ ), **the limit does not exist.**

### Exercise 3

Let  $z = 1 - i$ .

- (a) Write  $z$  in exponential form.

**Solution:**

We calculate the modulus  $r$  and argument  $\theta$ :

$$r = |z| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$
$$\theta = \tan^{-1}\left(\frac{-1}{1}\right) = -\frac{\pi}{4} \quad (\text{Quadrant IV})$$

Thus,  $z = \sqrt{2}e^{-i\pi/4}$ .

- (b) Compute  $z^{10}$  and express the answer in the form  $r(\cos(\theta) + i\sin(\theta))$ .

**Solution:**

Using the exponential form:

$$z^{10} = (\sqrt{2}e^{-i\pi/4})^{10} = (\sqrt{2})^{10}e^{-i10\pi/4}$$
$$= 2^5e^{-i5\pi/2} = 32e^{-i5\pi/2}$$

We simplify the angle  $-5\pi/2$ . Since  $-5\pi/2 = -2\pi - \pi/2$ , it is coterminal with  $-\pi/2$ .

$$z^{10} = 32e^{-i\pi/2}$$

In the required form:

$$z^{10} = 32\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$$

which simplifies to:

$$z^{10} = 32(0 - i) = -32i$$

## Exercise 4

Consider  $u(x, y) = x^2 - y^2 - 2y$ .

- (a) Verify that  $u(x, y)$  is harmonic.

**Solution:**

We compute the second partial derivatives:

$$u_x = 2x \implies u_{xx} = 2$$

$$u_y = -2y - 2 \implies u_{yy} = -2$$

$$\Delta u = u_{xx} + u_{yy} = 2 + (-2) = 0$$

Since the Laplacian is zero,  $u$  is harmonic.

- (b) Find the harmonic conjugate  $v(x, y)$  such that  $f = u + iv$  is analytic.

**Solution:**

We use the Cauchy-Riemann equations:  $u_x = v_y$  and  $u_y = -v_x$ . 1.  $v_y = u_x = 2x$ . Integrating with respect to  $y$ :

$$v = \int 2x \, dy = 2xy + h(x)$$

2. Differentiate  $v$  with respect to  $x$ :  $v_x = 2y + h'(x)$ . 3. Set  $v_x = -u_y$ . We know  $u_y = -2y - 2$ , so  $-u_y = 2y + 2$ .

$$2y + h'(x) = 2y + 2 \implies h'(x) = 2$$

$$h(x) = 2x + C$$

Thus,  $v(x, y) = 2xy + 2x + C$ . (We can take  $C = 0$ ).

- (c) **Bonus:** Write  $f(z)$  explicitly in terms of  $z$ .

**Solution:**

$$f(z) = u + iv = (x^2 - y^2 - 2y) + i(2xy + 2x)$$

$$f(z) = (x^2 - y^2 + 2ixy) + i(2x) - 2y$$

$$f(z) = (x + iy)^2 + 2ix - 2y$$

$$f(z) = z^2 + 2i(x + iy)$$

$$f(z) = z^2 + 2iz$$

## Exercise 5 - Graduate Only

**Proposition:** *If  $v$  is the harmonic conjugate of  $u$ , then  $-u$  is the harmonic conjugate of  $v$ .*

(a) Verify the previous statement.

**Solution:**

Let  $v$  be the harmonic conjugate of  $u$ . This implies two things:

- (a)  $u$  and  $v$  are harmonic.
- (b) They satisfy C-R:  $u_x = v_y$  and  $u_y = -v_x$ .

We want to check if  $-u$  is the harmonic conjugate of  $v$ . This requires:

- (a)  $v$  and  $-u$  are harmonic. (True, since  $u, v$  are harmonic).
- (b) They satisfy C-R, treating  $v$  as the real part and  $-u$  as the imaginary part. The C-R equations for the function  $g = v + i(-u)$  would be:

$$v_x \stackrel{?}{=} (-u)_y \quad \text{and} \quad v_y \stackrel{?}{=} -(-u)_x$$

Let's check the first condition:

$$(-u)_y = -u_y$$

From the original C-R ( $u_y = -v_x$ ), we have  $-u_y = v_x$ . So  $v_x = (-u)_y$  holds.

Let's check the second condition:

$$-(-u)_x = u_x$$

From the original C-R ( $u_x = v_y$ ), we have  $u_x = v_y$ . So  $v_y = -(-u)_x$  holds.

Since the C-R equations are satisfied for the pair  $(v, -u)$ ,  $-u$  is indeed the harmonic conjugate of  $v$ . (This corresponds to the fact that if  $f = u + iv$  is analytic, then  $-if = v - iu$  is also analytic).