Essential Taylor Series in Complex Analysis

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Below are the most frequently used Taylor series centered at $z_0 = 0$ and their regions of convergence. You can use them as building blocks to find the series expansions of more complicated functions through substitution, multiplication, differentiation, and integration.

1. The Exponential Function

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots \qquad (|z| < \infty) \tag{1}$$

2. Trigonometric Functions

$$\sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots \qquad (|z| < \infty) \tag{2}$$

$$\cos(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots \qquad (|z| < \infty) \tag{3}$$

3. Hyperbolic Functions

$$\sinh(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots \qquad (|z| < \infty) \tag{4}$$

$$\cosh(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \ldots \qquad (|z| < \infty) \tag{5}$$

4. The Geometric Series
(The most important tool for generating rational function series)

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \ldots \qquad (|z| < 1) \tag{6}$$

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n = 1 - z + z^2 - z^3 + \ldots \qquad (|z| < 1) \tag{7}$$

5. The Principal Logarithm
(Derived by integrating $\frac{1}{1+z}$)

$$\text{Log}(1+z) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{z^n}{n} = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots \qquad (|z| < 1) \tag{8}$$