Complex Analysis Worksheet
Chapter 1: Complex Numbers

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1. **Real and Imaginary Parts.**
   Consider the following complex functions:

   $$f(z) = \frac{1}{z}, \quad g(z) = \overline{z}, \quad h(z) = |z|, \quad k(z) = iz^2 \, .$$

   Express each function in the form $w(z) = u(x, y) + iv(x, y)$, where $z = x + iy$ and $u, v : \mathbb{R}^2 \to \mathbb{R}$ are real-valued functions of two real variables.

2. **Mappings.**
   Sketch the images of the following sets under the given transformations:

   (a) The image of the square with vertices at 0, 1, i, and $1 + i$ under the mapping:

   $$f(z) = z + 1 + i$$

   (b) The image of the same square under the mapping:

   $$g(z) = \overline{z}$$

   (c) The image of the line segment with endpoints 0 and $1 + i$ under the mapping:

   $$h(z) = ze^{i\frac{\pi}{4}}$$

3. **Limit Definitions.**
   Consider the following definitions for limits involving infinity:

   (a) *We say that $\lim_{z \to z_0} f(z) = \infty$ if for every $M > 0$ there exists a $\delta > 0$ such that $0 < |z - z_0| < \delta$ implies $|f(z)| > M$.*

   (b) *We say that $\lim_{z \to \infty} f(z) = w_0$ if for every $\epsilon > 0$ there exists an $L > 0$ such that $|z| > L$ implies $|f(z) - w_0| < \epsilon$.*

   Using these definitions, prove the following equivalences:

(a) $\lim_{z\to\infty} f(z) = w_0$ if and only if $\lim_{z\to 0} f\left(\frac{1}{z}\right) = w_0$.

(b) $\lim_{z\to z_0} f(z) = \infty$ if and only if $\lim_{z\to z_0} \frac{1}{f(z)} = 0$.

4. Computing Limits.

Evaluate the following limits. If a limit is infinite, state that it goes to ∞.

(a) $\lim_{z\to i} \dfrac{4z^2}{z-i}$

(b) $\lim_{z\to 0} \dfrac{z^2+1}{|z|^2}$

(c) $\lim_{z\to i} \dfrac{z^2+1}{z-i}$

5. Limit Non-Existence.

Show that the following limit does not exist:

$$\lim_{z\to 0} e^{1/z}$$

Hint: Compute the limit along the real axis and the imaginary axis separately, and compare the results.