Complex Analysis Worksheet

Chapter 2: Analytic Functions

Guido Mazzuca

02/06/2026

---

1. Real and Imaginary Parts.

For each of the following functions $f(z)$, find the real part $u(x, y)$ and the imaginary part $v(x, y)$ such that $f(z) = u(x, y) + iv(x, y)$.

(a) $f(z) = z^3 + z + 1$

(b) $f(z) = \dfrac{1}{\bar{z}}$

(c) $f(z) = e^{-z}$

2. Complex Limits.

Evaluate the following limits. If the limit does not exist, explain why.

(a) $\lim_{z \to 2i} (z^2 + 4z)$

(b) $\lim_{z \to 1+i} \dfrac{z^2 - 2z + 2}{z - (1 + i)}$

(c) $\lim_{z \to 0} \dfrac{\bar{z}}{z}$

3. Differentiation by Definition.

Using the limit definition of the derivative:

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

compute the derivative of the function $f(z) = 3z^2 - iz$.

4. Checking Differentiability.

Use the Cauchy-Riemann equations to determine the set of points z in the complex plane where the derivative $f'(z)$ exists.

(a) $f(z) = e^x(\cos y + i \sin y)$

(b) $f(z) = x^2 - iy^2$

State the domain (if any) where each function is analytic.

5. Harmonic Conjugates.
Consider the function $u(x, y) = x^3 - 3xy^2$.

(a) Verify that $u(x, y)$ is harmonic in the entire plane.

(b) Find the harmonic conjugate $v(x, y)$.

(c) Write the resulting analytic function $f(z) = u + iv$ explicitly in terms of z.

6. Properties of Analytic Functions.
Let $f(z) = u(x, y)+iv(x, y)$ be an analytic function in a domain D. Prove that if the real part $u(x, y)$ is constant throughout D, then $f(z)$ must be constant throughout D.
Hint: Use the Cauchy-Riemann equations to show that $f'(z) = 0$ everywhere in D.

7. Laplacian Identity.
Let $f(z)$ be an analytic function in a domain D. Show that the following identity holds:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^2 = 4|f'(z)|^2$$

Hint: Start by writing $|f(z)|^2 = f(z)\overline{f(z)}$.