Complex Analysis Worksheet

Chapter 3: Elementary Functions

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1. The Exponential Function.
Find all complex values of z that satisfy the following equations:

(a) $e^z = -1 + i\sqrt{3}$

(b) $e^{2z-1} = 1$

2. Principal Logarithms and Complex Powers.
Compute the principal values of the following expressions. Write your final answers in the algebraic (or rectangular) form $x + iy$:

(a) $\text{Log}(-ei)$

(b) i^{-i}

(c) $(1-i)^{4i}$

3. Branch Cuts and Evaluation.
Let $f(z) = \log(\sqrt{3}z + 1)$ be the branch of the multiple-valued logarithmic function defined by using the standard branch-cut

(a) Specify the branch cut for this function. Which ray is removed from the complex plane?

(b) Evaluate $f(-i)$ using this specific branch.

4. Branch Cuts and Evaluation.
Consider the multi-valued functions $f(z) = (z)^{1/2}(z-1)^{1/2}$, $g(z) = (z(z-1))^{1/2}$.

(a) Show that the principal values of $f(z) \neq g(z)$.

(b) Show that for all $x \in (0,1)$

$$\lim_{\varepsilon \to 0^+} f(x + i\varepsilon) = -\lim_{\varepsilon \to 0^+} f(x - i\varepsilon)\,.$$

(c) **Bonus.** Show that if $x \in (-\infty, 0)$ then

$$\lim_{\varepsilon \to 0^+} f(x + i\varepsilon) = \lim_{\varepsilon \to 0^+} f(x - i\varepsilon) \,.$$

Does it mean that the branch cut of $f(z)$ might be different than what we expect?

5. Sketching Branch Cuts.
For each of the following functions, determine the location of the branch cut(s) in the complex plane assuming we are using the principal branch of the logarithm, $\text{Log}(w)$ (where the cut for w is the negative real axis $(-\infty, 0]$). Sketch the branch cuts for z.

(a) $f(z) = \text{Log}(z - 2 + i)$

(b) $g(z) = \text{Log}\left(\frac{z-1}{z+1}\right)$

6. Complex Trigonometric Functions.
Using the definitions of complex sine and cosine in terms of the exponential function:

(a) Prove the identity $\sin^2 z + \cos^2 z = 1$ for all $z \in \mathbb{C}$.

(b) Show that $|\sinh z|^2 = \sinh^2 x + \sin^2 y$.

7. Inverse Hyperbolic Functions.
Consider the inverse hyperbolic tangent function, $w = \text{arctanh}(z)$, which is defined implicitly as the inverse of $z = \tanh(w)$.

(a) By writing $\tanh(w)$ in terms of complex exponentials (e^w and e^{-w}) and solving for w, derive the logarithmic formula:

$$\text{arctanh}(z) = \frac{1}{2} \log\left(\frac{1+z}{1-z}\right)$$

(b) Let $\text{Arctanh}(z)$ denote the principal branch of this function, obtained by using the principal branch of the logarithm (Log). Determine the exact branch cut(s) for $\text{Arctanh}(z)$ in the complex plane and sketch them.
Hint: For what values of z is the argument of the logarithm a negative real number?