Complex Analysis Worksheet

Chapter 5: Liouville's Theorem, Taylor, and Laurent Series

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Undergraduate Exercises

1. **Determining a Function from its Behavior.**
   Suppose $f(z)$ is analytic everywhere in the complex plane except for an isolated singularity at $z = 0$. Furthermore, assume that $f(z)$ behaves like $\frac{2}{z}$ as $z \to 0$, and that $\lim_{z\to\infty} f(z) = 0$. Use Liouville's Theorem to determine the exact form of $f(z)$.

2. **Laurent Series in Concentric Domains.**
   Consider the rational function:

$$f(z) = \frac{1}{(z-1)(z-3)}$$

   Find the Laurent series expansion for $f(z)$ centered at $z_0 = 0$ in each of the following domains. Write your answers using summation notation.

   (a) The open disk $|z| < 1$.
   (b) The annulus $1 < |z| < 3$.
   (c) The exterior region $|z| > 3$.

   Hint: Begin by finding the partial fraction decomposition of $f(z)$.

3. **Laurent Series around a Singularity.**
   Find the Laurent series expansion of the function:

$$f(z) = \frac{1}{z^2 - 4}$$

   valid in the punctured disk (annulus) $0 < |z - 2| < 4$.
   Hint: You are expanding around $z_0 = 2$. Let $w = z - 2$, rewrite the function in terms of w, and expand using the geometric series before substituting back.

4. **Manipulating Standard Series.**
Using the known Taylor series for $e^z, \sin(z)$, and $\cos(z)$, find the Laurent or Taylor series for the following functions centered at $z_0 = 0$:

(a) $f(z) = z^2 \cos\left(\frac{1}{z}\right)$ (valid for $|z| > 0$)

(b) $g(z) = \frac{1 - e^{-z^2}}{z}$ (valid for $|z| > 0$)

5. **Evaluating Integrals using Series.**
Let C be the unit circle $|z| = 1$ traversed counterclockwise; compute the following integrals by first expanding the function inside the integral as a Laurent series around $z_0 = 0$ and then applying the Cauchy Integral Formula:

(a) $\oint_C z^3 e^{2/z}\, dz$

(b) $\oint_C \frac{\sin(z)}{z^6}\, dz$

Graduate Exercises

6. **Generalized Liouville's Theorem.**
Let $f(z)$ be an entire function. Suppose there exist real constants $C > 0$ and $R > 0$, and a non-negative integer N, such that:

$$|f(z)| \le C|z|^N \quad \text{for all } |z| > R$$

Prove that $f(z)$ must be a polynomial of degree at most N.

Hint: Write $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Use Cauchy's Inequality to bound the coefficients $a_k = \frac{f^{(k)}(0)}{k!}$ by integrating over a circle of radius $r > R$. What happens to the bound for a_k when $k > N$ and we let $r \to \infty$?