Complex Analysis

Worksheet: Series Convergence and Residues

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Undergraduate Exercises

Exercise 1 (Series by Integration and Convergence)**.** Consider the geometric series for the function $f(z) = \dfrac{1}{1+z}$.

(a) Write down the Maclaurin series for $f(z)$. State its radius of convergence R, and explicitly describe the largest domain where the series converges **uniformly**.

(b) Does the series for $f(z)$ converge anywhere on the boundary $|z| = R$? Justify your answer.

(c) By integrating the series term-by-term from 0 to z, find the Maclaurin series for the principal branch of the logarithm, $\mathrm{Log}(1+z)$.

(d) What is the radius of convergence and the domain of uniform convergence for your new integrated series? Does the integrated series converge on the boundary?
Hint for (d): Consider the Alternating Series Test or Dirichlet's Test for boundary points, being careful at $z = -1$.

Exercise 2 (Series by Differentiation and Convergence)**.** Consider the geometric series for the function $g(z) = \dfrac{1}{1-z}$.

(a) Write down the Maclaurin series for $g(z)$. State its radius of convergence R and its domain of uniform convergence. Does it converge anywhere on the boundary $|z| = R$?

(b) Differentiate the series term-by-term to find the Maclaurin series for $h(z) = \dfrac{1}{(1-z)^2}$.

(c) State the radius of convergence and the domain of uniform convergence for the differentiated series. Check the convergence of this new series on the boundary $|z| = R$. How does differentiation affect boundary convergence compared to integration?

Exercise 3 (Residues via the Simple Pole Limit Formula)**.** Find all isolated singularities of the following function, classify them (which is the order of the pole), and compute the residue at each singularity using the limit formula for simple poles ($\text{Res}(f, z_0) = \lim_{z \to z_0}(z - z_0)f(z)$) or the $P(z_0)/Q'(z_0)$ rule:

$$f(z) = \frac{e^{iz}}{z^2 + 4}$$

Exercise 4 (Residues via the Higher-Order Pole Formula)**.** Consider the function:

$$f(z) = \frac{z^2 + 1}{(z - i)^3}$$

Identify the pole and its order m. Compute the residue at this pole using the standard derivative limit formula for poles of order m:

$$\text{Res}(f, z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z) \right]$$

Exercise 5 (Residues via Laurent Series Expansion)**.** Some singularities are not poles, meaning the limit formulas cannot be used. Consider the function:

$$f(z) = z^3 \sin\left(\frac{1}{z}\right)$$

(a) What type of singularity does $f(z)$ have at $z = 0$?

(b) Compute the residue at $z = 0$ by expanding the function into its Laurent series and directly identifying the coefficient a_{-1}.

Graduate Exercises

Exercise 6 (Summing Series via the Residue Theorem)**.** Use the Residue Theorem to exactly evaluate the infinite sum:

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + 1}$$

Steps to follow:

(a) Consider the meromorphic function $f(z) = \dfrac{\pi \cot(\pi z)}{z^2 + 1}$. Find all poles of $f(z)$ in the complex plane and compute the residue at each pole.

(b) Consider a large square contour C_N centered at the origin with vertices at $\pm(N + \frac{1}{2}) \pm i(N + \frac{1}{2})$ for an integer N. Apply the Residue Theorem to $\oint_{C_N} f(z)\, dz$.

(c) Assuming that $\lim_{N\to\infty} \oint_{C_N} f(z)\,dz = 0$ (because $\cot(\pi z)$ is uniformly bounded on these squares), deduce the exact value of the infinite sum.

(d) *Bonus:* Using your result from (c) and the symmetry of the sum, deduce the value of $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$.

Exercise 7. Let $f(z)$ be a analytic in a simple connected domain D and let $z_0 \in D$ be the only zero of $f(z)$ in D. Prove that if γ is a simple and closed curve in D that winds once around z_0 then

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz = m$$

where m is the order of the zero of $f(z)$ at z_0.