Worksheet Solutions: The Residue Theorem and Real Integrals

Instructor: Guido Mazzuca

Suggested Point Distribution:
Undergraduate Total: 80 pts | Graduate Total: 100 pts

- **Exercise 1:** 20 points (5 pts for singularities, 15 pts for residues)
- **Exercise 2:** 20 points (5 pts per step)
- **Exercise 3:** 20 points (15 pts for residues/calc, 5 pts for contour bounding)
- **Exercise 4:** 20 points
- **Exercise 5 (Grad):** 20 points (5 pts per part)

Undergraduate Exercises

Exercise 1 (Warm-up: Residue Calculation)**.** Consider $f(z) = \frac{z^2+2}{(z-1)^2(z^2+4)}$. Find all singularities and their residues.

Solution: **(a) Singularities:** The denominator factors as $(z-1)^2(z-2i)(z+2i)$. The roots of the numerator do not cancel any of these. Thus, the singularities are:

- $z = 1$: Pole of order 2
- $z = 2i$: Simple pole (order 1)
- $z = -2i$: Simple pole (order 1)

(b) Residues:
At $z = 2i$: Let $P(z) = z^2 + 2$ and $Q(z) = (z-1)^2(z^2+4)$. Then $Q'(z) = 2(z-1)(z^2+4) + (z-1)^2(2z)$.

$$\text{Res}(f, 2i) = \frac{P(2i)}{Q'(2i)} = \frac{-4+2}{0 + (2i-1)^2(4i)} = \frac{-2}{(-3-4i)(4i)} = \frac{-2}{16-12i} = \frac{-1}{8-6i}$$

Multiply by the conjugate: $\frac{-1(8+6i)}{64+36} = \frac{-8-6i}{100} = -\frac{2}{25} - i\frac{3}{50}$.

At $z = -2i$: Since the polynomial coefficients are entirely real, the residue at the conjugate pole is the conjugate of the residue.

$$\text{Res}(f, -2i) = -\frac{2}{25} + i\frac{3}{50}$$

At $z = 1$: Use the derivative formula for a pole of order $m = 2$:

$$\text{Res}(f, 1) = \lim_{z\to1} \frac{d}{dz}\left[\frac{z^2+2}{z^2+4}\right] = \lim_{z\to1} \frac{2z(z^2+4) - (z^2+2)(2z)}{(z^2+4)^2} = \frac{2(5) - (3)(2)}{25} = \frac{4}{25}$$

(Self-check: The sum of the residues is $-4/25 + 4/25 = 0$, which matches the fact that $f(z) \sim 1/z^2$ for large z.)

Exercise 2. Evaluate $\displaystyle\int_0^{2\pi} \frac{1}{5 - 4\cos(\theta)}\, d\theta$.

Solution: **(a) Substitution:** $z = e^{i\theta}$. Then $\cos(\theta) = \frac{z+z^{-1}}{2}$ and $d\theta = \frac{dz}{iz}$.

(b) Contour Integral:

$$\oint_{|z|=1} \frac{1}{5 - 4\left(\frac{z+z^{-1}}{2}\right)} \frac{dz}{iz} = \oint_{|z|=1} \frac{dz}{i(5z - 2z^2 - 2)} = \oint_{|z|=1} \frac{i\, dz}{2z^2 - 5z + 2}$$

(c) Finding Poles: Factor the denominator $2z^2 - 5z + 2 = (2z-1)(z-2)$. The roots are $z = 1/2$ and $z = 2$. Only the simple pole at $z = 1/2$ lies *inside* the unit circle.

(d) Evaluate: Let $f(z) = \frac{i}{2z^2-5z+2}$. The residue at $z = 1/2$ is:

$$\text{Res}(f, 1/2) = \left.\frac{i}{\frac{d}{dz}(2z^2 - 5z + 2)}\right|_{z=1/2} = \left.\frac{i}{4z-5}\right|_{z=1/2} = \frac{i}{2-5} = -\frac{i}{3}$$

By the Residue Theorem, the integral is $2\pi i\left(-\frac{i}{3}\right) = \frac{2\pi}{3}$.

Exercise 3 (Improper Integrals of Rational Functions)**.** Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{x^4+16}\, dx$.

Solution: Let $f(z) = \frac{z^2}{z^4+16}$. We integrate over a large semicircular contour C in the upper half-plane (UHP), bounded by $[-R, R]$ and the arc C_R.

Poles in UHP: We solve $z^4 = -16 = 16e^{i\pi}$. The roots are $z_k = 2e^{i(\pi/4 + k\pi/2)}$. The two roots in the UHP ($k = 0, 1$) are:

$$z_0 = 2e^{i\pi/4} = \sqrt{2} + i\sqrt{2} \quad \text{and} \quad z_1 = 2e^{i3\pi/4} = -\sqrt{2} + i\sqrt{2}$$

Residues: Using the $P(z)/Q'(z)$ rule: $\text{Res}(f, z_k) = \frac{z_k^2}{4z_k^3} = \frac{1}{4z_k}$.

$$\text{Res}(f, z_0) = \frac{1}{4(2e^{i\pi/4})} = \frac{1}{8}e^{-i\pi/4} = \frac{\sqrt{2}}{16} - i\frac{\sqrt{2}}{16}$$

$$\text{Res}(f, z_1) = \frac{1}{4(2e^{i3\pi/4})} = \frac{1}{8}e^{-i3\pi/4} = -\frac{\sqrt{2}}{16} - i\frac{\sqrt{2}}{16}$$

Sum of residues $= -i\frac{2\sqrt{2}}{16} = -i\frac{\sqrt{2}}{8}$.

Integral Evaluation:

$$\oint_C f(z)\, dz = 2\pi i\left(-i\frac{\sqrt{2}}{8}\right) = \frac{\pi\sqrt{2}}{4}$$

On C_R, $|f(z)| \approx \frac{R^2}{R^4} = \frac{1}{R^2}$. By the ML-inequality, $\left|\int_{C_R} f(z)dz\right| \leq \frac{\pi R}{R^2} = \frac{\pi}{R} \to 0$ as $R \to \infty$. Thus, the integral over the real line evaluates precisely to $\frac{\pi\sqrt{2}}{4}$.

Exercise 4 (Integrals involving Sine and Cosine)**.** Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(2x)}{x^2+2x+5}\,dx$.

Solution: As hinted, we integrate the complex function $f(z) = \frac{e^{2iz}}{z^2+2z+5}$ over the standard semicircular contour in the UHP.

Poles: $z^2+2z+5=0 \implies (z+1)^2 = -4 \implies z = -1 \pm 2i$. The only pole in the UHP is $z_0 = -1+2i$.

Residue:

$$\text{Res}(f, -1+2i) = \frac{e^{2iz}}{2z+2}\bigg|_{z=-1+2i} = \frac{e^{2i(-1+2i)}}{2(-1+2i)+2} = \frac{e^{-4-2i}}{4i}$$

Integral Evaluation: By the Residue Theorem:

$$\oint_C f(z)\,dz = 2\pi i\left(\frac{e^{-4}e^{-2i}}{4i}\right) = \frac{\pi}{2e^4}(\cos(-2) + i\sin(-2)) = \frac{\pi}{2e^4}(\cos 2 - i\sin 2)$$

The integral over C_R vanishes by Jordan's Lemma (since we have e^{iaz} with $a = 2 > 0$ in the UHP). Thus:

$$\int_{-\infty}^{\infty} \frac{e^{2ix}}{x^2+2x+5}\,dx = \frac{\pi\cos 2}{2e^4} - i\frac{\pi\sin 2}{2e^4}$$

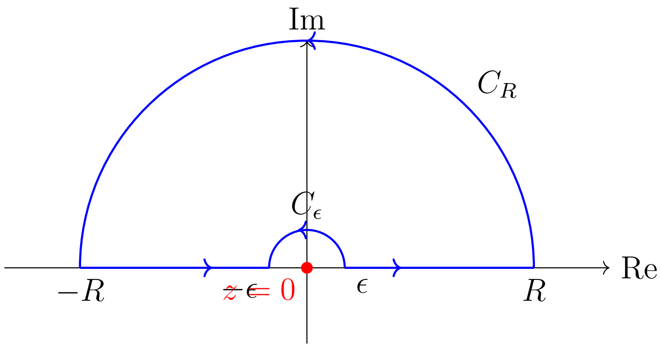
Taking the real part yields our final answer: $\frac{\pi\cos 2}{2e^4}$.

Graduate Exercises

Exercise 5 (The Dirichlet Integral and Indented Contours)**.** Evaluate $\displaystyle\int_0^{\infty} \frac{\sin(x)}{x}\,dx$.

Solution: **(a)** The function $f(z) = \frac{e^{iz}}{z}$ has a simple pole at $z=0$. Integrating straight through a pole on the real axis results in a divergent, undefined integral in standard Cauchy theory.

(b) The indented contour bypasses the origin. No poles lie strictly *inside* this contour.



(c) Since $f(z)$ is analytic everywhere inside the closed contour, Cauchy's Theorem gives:

$$\oint f(z)\,dz = 0$$

(d) Breaking the contour into its four segments:

$$\int_{-R}^{-\epsilon} \frac{e^{ix}}{x}\,dx + \int_{C_\epsilon} \frac{e^{iz}}{z}\,dz + \int_{\epsilon}^{R} \frac{e^{ix}}{x}\,dx + \int_{C_R} \frac{e^{iz}}{z}\,dz = 0$$

As $R \to \infty$, the C_R integral goes to 0 (Jordan's Lemma).
As $\epsilon \to 0$, the C_ϵ integral (traversed clockwise, spanning π radians) evaluates to $-i\pi\,\text{Res}(f, 0) = -i\pi(e^{i(0)}) = -i\pi$.

Combine the two real segments. Let $x = -t$ in the first integral:

$$\int_{-R}^{-\epsilon} \frac{e^{ix}}{x}\,dx = \int_{R}^{\epsilon} \frac{e^{-it}}{-t}(-dt) = -\int_{\epsilon}^{R} \frac{e^{-ix}}{x}\,dx$$

Add this to the third integral:

$$\int_{\epsilon}^{R} \frac{e^{ix} - e^{-ix}}{x}\,dx = \int_{\epsilon}^{R} \frac{2i\sin(x)}{x}\,dx$$

Substitute these limits back into the contour equation:

$$2i\int_0^\infty \frac{\sin(x)}{x}\,dx - i\pi + 0 = 0 \implies 2i\int_0^\infty \frac{\sin(x)}{x}\,dx = i\pi$$

Divide by $2i$:

$$\int_0^\infty \frac{\sin(x)}{x}\,dx = \frac{\pi}{2}$$